ALGEBRAIC GRAPHS FOR INTERPRETATION

With Statistics

by

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PREFACE

This book covers algebraic graphs in the same way as its predecessor, *Graphs for Interpretation*, covered arithmetical graphs Its aims are

(1) To interest and train pupils in interpreting algebraic graphs by providing many of these graphs already drawn
(2) To provide a natural sequence which will simplify progress from type to type
(3) To introduce at appropriate stages such essential features as the graph of a function, the equation of a graph, determination of a linear law from a graph, axis of symmetry, turning points, etc
(4) To give the pupils just sufficient practice in drawing their own graphs to ensure that they can apply the knowledge gained

This book covers the graphical work in Algebra required for the Scottish Certificate of Education (Ordinary and Higher Grades) and for the English Certificate of Education to beyond the Ordinary Level

Chapters 8, 9, and 10 include a short introduction to Statistics which it is hoped will prove of benefit to pupils who intend to enter for Arithmetic in the Scottish Certificate of Education examination

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G L B

CONTENTS

Chapter 1

INTRODUCTION

Co-ordinates

If, from a base camp on the Antarctic continent, an explorer sets up four sub-stations, their locations could be described from the base as follows

A sub-station	3 miles east and 2 miles north
B sub-station	$2\frac{1}{2}$ miles west and 3 miles north
C sub-station	2 miles west and 4 miles south
D sub-station	2 miles east and $2\frac{1}{2}$ miles south

In order to specify the positions of these sub-stations, we have used the base camp as a starting point or **origin** and two lines at right angles to one another crossing at the origin, the N–S line and the E–W line, as **axes of reference**

Fig 1 shows the four sub-stations, and the axes crossing at O and dividing the area into four parts called **quadrants.**

A is in the 1st quadrant
B is in the 2nd quadrant.
C is in the 3rd quadrant.
D is in the 4th quadrant.

The figures marked along the axes represent measurements in miles

In the same way the position of a point on a plane can be described from two perpendicular axes X′OX, YOY′ crossing at the origin O (See Fig 2) X′OX is called the x-axis and YOY′ is called the y-axis.

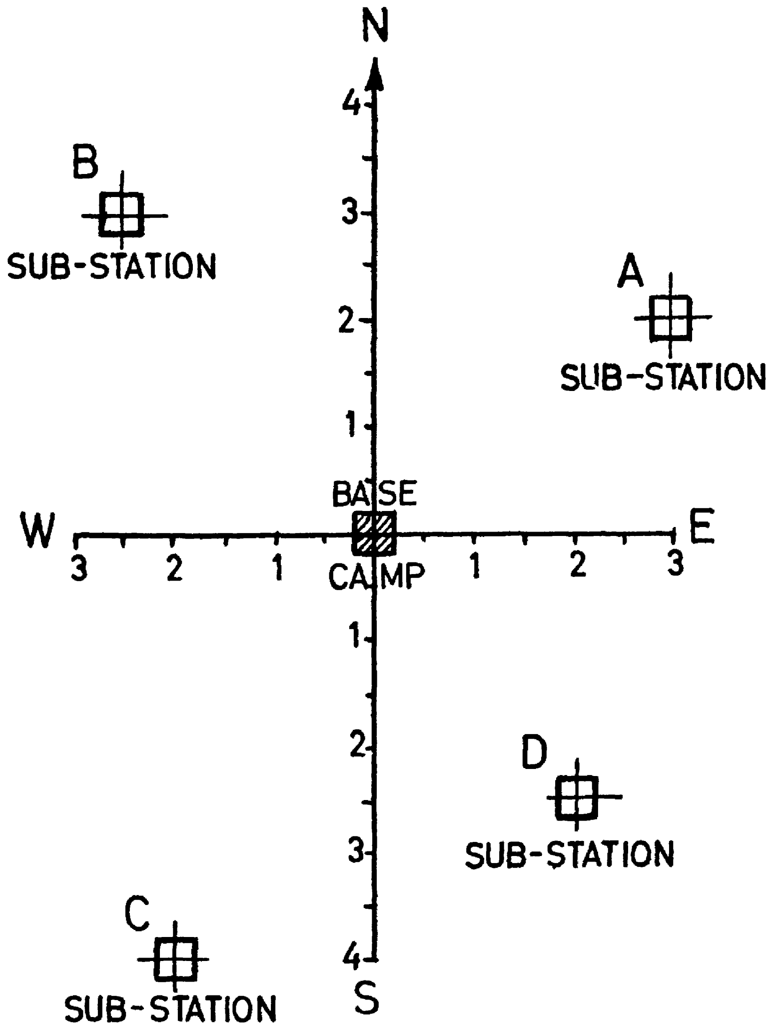
Start from O and move 3 units in the direction OX to point M, and then 2 units in the direction of OY to point A

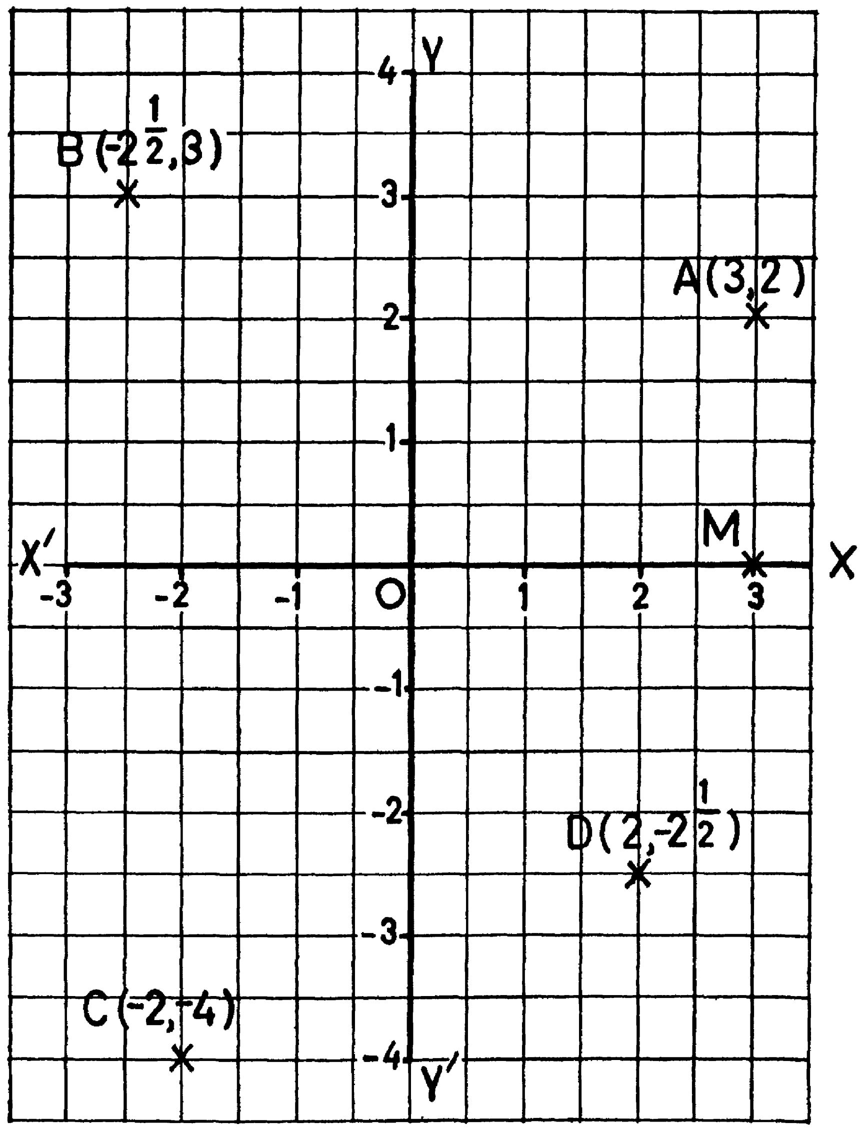
OM = 3 units is called the x-co-ordinate of point A
MA = 2 units is called the y-co-ordinate of point A
Point A is described as (3, 2).

Distances along X′OX to the right of O are reckoned positive and to the left of O are reckoned negative. Similarly, distances in the direction OY are reckoned positive and in the direction OY′ are reckoned negative

Thus, B is the point $(-2\frac{1}{2}, 3)$
C is the point $(-2, -4)$
D is the point $(2, -2\frac{1}{2})$

If each unit represents 1 mile we have described the position of sub-stations A, B, C, D by means of **cartesian co-ordinates** (named after René Descartes who first used this method of fixing the position of a point).


N
E
S
W
BASE
CAMP
A
B
C
D
SUB-STATION
SUB-STATION
SUB-STATION
SUB-STATION
Fig. 1


Y
Y′
X
X′
O
M
A(3,2)
B(-2 1/2,3)
C(-2,-4)
D(2,-2 1/2)
Fig. 2

Notes. 1. The x-co-ordinate, sometimes called the **abscissa,** is always written first. The y-co-ordinate is known as the **ordinate.**

2 Points may be marked (or plotted) either by two thin lines forming a cross, or by a tiny dot surrounded by a circle

Exercises A

1. Read off the co-ordinates of the points E, F, G, H, K, L, M, N, P, R, S, marked on Fig 3

(*a*) Say in which quadrants both co-ordinates are (i) positive, (ii) negative
(*b*) Say in which quadrant the x-co-ordinate is positive and the y-co-ordinate negative
(*c*) Say in which quadrant the x-co-ordinate is negative and the y-co-ordinate positive
(*d*) What is the value of the y-co-ordinate for all points on the x-axis?
(*e*) What is the value of the x-co-ordinate for all points on the y-axis?

2. On your graph books draw the x and y axes crossing centrally at O and with 1 inch for 1 unit along each axis mark (or plot) the following points

K (1, 3), L (−3, 2), M (2, 0), N (−1, −2), P (3, −1),
Q (0, 3), R (−3, 0), S (2, −4), T (0, −1), V (−1, 1)

Find by measurement the lengths of KL, MN, PQ, RS, and TV

3. With axes as in Exercise **2**, plot the following points

A ($2\frac{1}{2}$, 2), B ($1\frac{1}{2}$, −3), C (−3, 1), D (3, −4),
E (−2, −1), F (−2, 4), G (3, $3\frac{1}{2}$), H (−2, −2)

Find the co-ordinates of the middle points of the lines AB, CD, EF, GH

4. In Fig 4 find the co-ordinates of the corners of rectangle ABCD

(*a*) Write down the x-co-ordinate of any other point on BC
(*b*) Write down the x-co-ordinate of any other point on DA
(*c*) Write down the y-co-ordinate of any other point on AB
(*d*) Write down the y-co-ordinate of any other point on CD
(*e*) Find the co-ordinates of the mid-point of AC Compare this with the average of

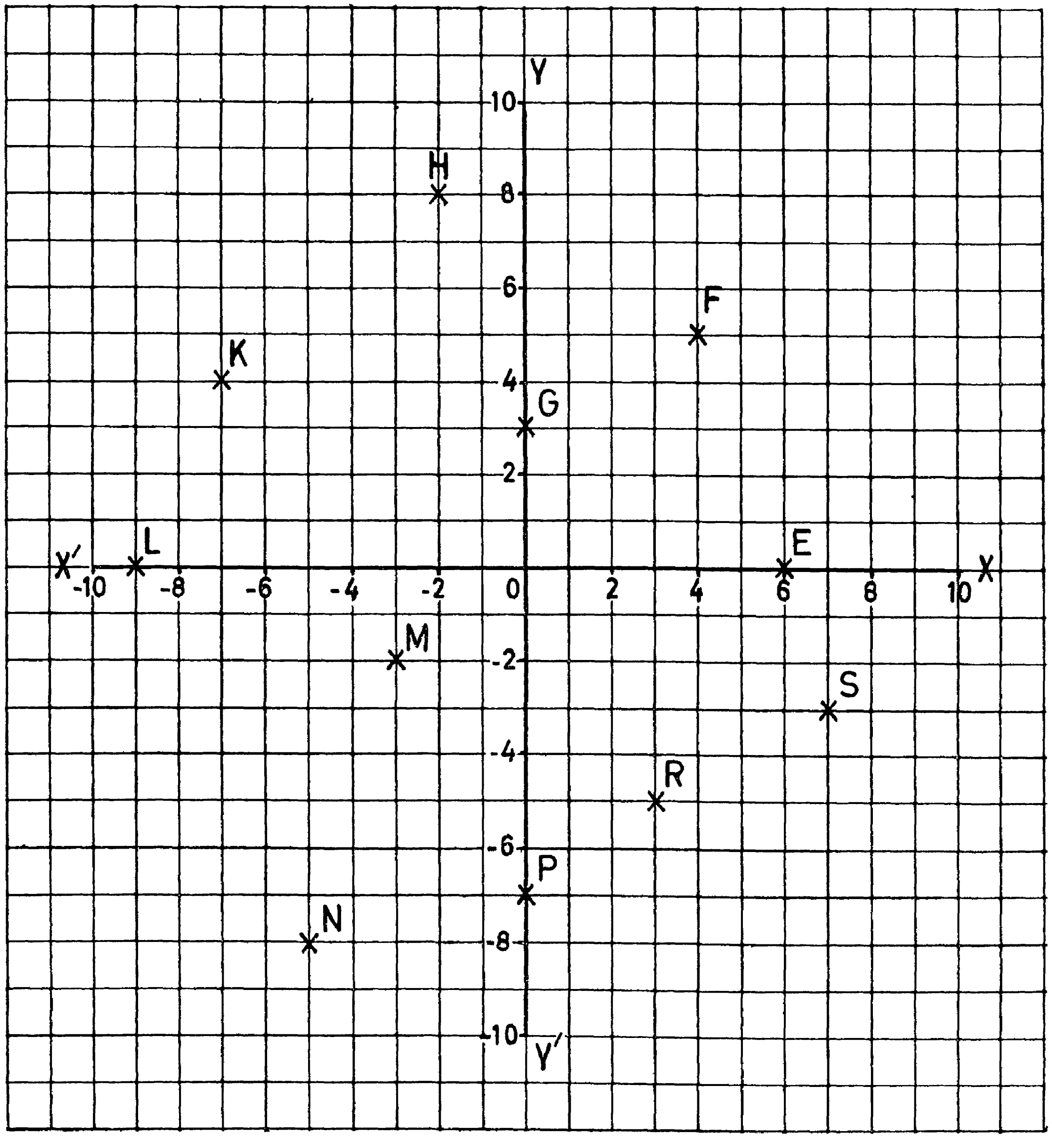
(i) The x-co-ordinates of A and C
(ii) The y-co-ordinates of A and C

5. By plotting the following points, show that each set lies on a straight line

(*a*) (10, 9), (3, 0), (−4, −9)
(*b*) (4, −2), (2, $\frac{1}{2}$), (−2, $5\frac{1}{2}$)

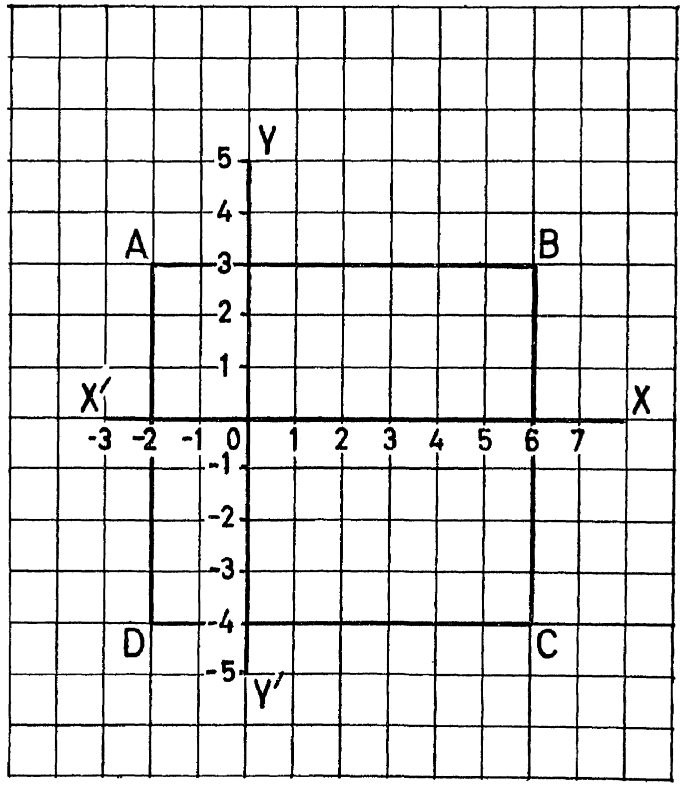
6. Plot the points (0, 1), (−9, 1), and (−9, −11), and show that they form a triangle whose area is 54 square units

7. By plotting the points A (1, 2), B (−1, −1), C (2, −3), show that they form a right-angled isosceles △ ABC Measure the lengths of its sides AB and BC



Y
10
H
8
6
F
K
4
G
2
X′
L
E
X
-10
-8
-6
-4
-2
0
2
4
6
8
10
M
-2
S
-4
R
-6
P
N
-8
-10
Y′

Fig 3



Y
5
4
A
B
3
2
1
X′
X
-3
-2
-1
0
1
2
3
4
5
6
7
-1
-2
-3
-4
D
C
-5
Y′

Fig 4

8. Plot the following points and show by measurement that they are all the same distance from the origin

$$(4, 3),\ (-3, 4),\ (0, 5),\ (-4, -3),\ (3, -4),\ (-5, 0)$$

Which kind of smooth curve would pass through all these points?

9. Plot the following points and show by measurement that they are all the same distance from the point C (3, 2)·

$$(9, 10),\ (3, 12),\ (-5, 8),\ (-3, -6),\ (11, -4)$$

Draw the smooth curve to pass through these points.

10. Find by measurement the perimeter of the triangle whose vertices are

$$P\ (-2, 2),\ Q\ (-1, -2),\ R\ (2, -1)$$

Functions of x

Consider

(a) $3x + 1$ (b) $x^2 + 4x - 3$. (c) $x^3 - 6x + 2$.

Each of these expressions contains the variable x and its value depends on the value of x. Such expressions are known as functions of x, written $f(x)$ or $F(x)$ or sometimes $G(x)$, etc (*N B* f is short for "a function of" and does NOT mean that (x) is to be multiplied by f)

We may rewrite the expressions as

$f(x) = 3x + 1$	a first-degree or **linear function.**
$G(x) = x^2 + 4x - 3$	a second-degree or **quadratic function.**
$F(x) = x^3 - 6x + 2$	a third-degree or **cubic function.**

They are, respectively, first-degree, second-degree, and third-degree functions of x, according to the highest power of x in each

Here, the value of the function $3x + 1$ when $x = 1$ is denoted by $f(1)$, when $x = -2$ by $f(-2)$, and so on, thus

$$\begin{aligned} f(x) &= 3x + 1 \\ f(1) &= 3 \cdot 1 + 1 = +4 \\ f(0) &= 3 \cdot 0 + 1 = +1 \\ f(-1) &= 3 \cdot -1 + 1 = -2 \\ f(-2) &= 3 \cdot -2 + 1 = -5 \end{aligned}$$

Similarly, if $G(x) = x^2 + 4x - 3$

$$\begin{aligned} G(3) &= 3^2 + 4 \cdot 3 - 3 = 18 \\ G(-1) &= (-1)^2 + 4 \cdot -1 - 3 = -6 \\ G(0) &= (0)^2 + 4 \cdot 0 - 3 = -3 \end{aligned}$$

Exercises B

1. If $f(x) = 3x + 1$, find $f(2)$, $f(5)$, $f(9)$, $f(-3)$, $f(-6)$

2. If $G(x) = x^2 + 4x - 3$, find $G(7)$, $G(6)$, $G(1)$, $G(-2)$, $G(-3)$, $G(-4)$, $G(-5)$.

Graph of a Function

If instead of $f(x) = 3x + 1$ we write $y = 3x + 1$, then y is a function of x, and corresponding to

$$x = -2, -1, 0, 1$$

we have

$$y = -5, -2, 1, 4,$$

respectively.

If we now plot the points $(-2, -5)$, etc., and join them as shown in Fig. 5, we arrive at a line. This line is called either

(*a*) the graph of the function $3x + 1$,

or (*b*) the graph of the equation $y = 3x + 1$.

Similarly, by varying the value of x in the function $x^2 + 4x - 3$ we find a value of this function corresponding to each value of x. The table compiled from previous calculations is

$$\begin{aligned} x &= -5, -4, -3, -2, -1, \quad 0, 1. \\ y &= \quad 2, -3, -6, -7, -6, -3, 2. \end{aligned}$$

Proceeding as before, we obtain the curved line shown in Fig. 6. This line is called

(*a*) the graph of the function $x^2 + 4x - 3$,

or (*b*) the graph of the equation $y = x^2 + 4x - 3$

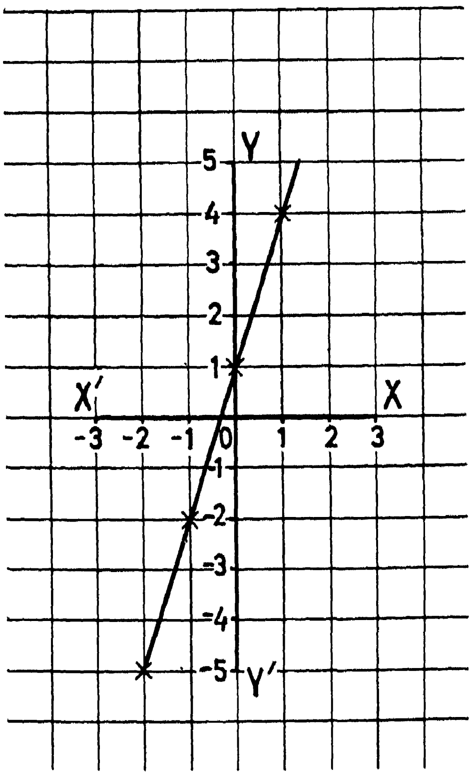


Fig. 5

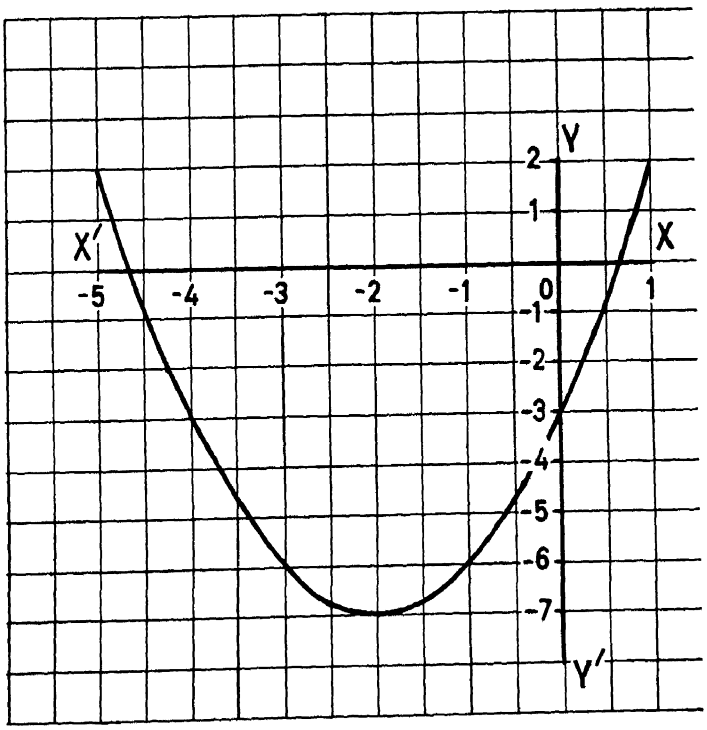


Fig. 6

Example

Draw the graph of the function $x^3 - 6x + 2$ from $x = -3$ to $x = 3$

Firstly we find the value of the function when $x = -3$, $x = -2$, and so on, thus

If $F(x) = x^3 - 6x + 2$

$$F(-3) = (-3)^3 - 6 \quad -3 + 2 = -27 + 18 + 2 = -7$$
$$F(-2) = (-2)^3 - 6 \quad -2 + 2 = -8 + 12 + 2 = +6$$

and so on

(The pupil should work out $F(-1)$, $F(0)$, $F(1)$, $F(2)$, $F(3)$, and compare his answers with those in the table following)

Now let $y = x^3 - 6x + 2$ and our results are as follows

$$x = -3, -2, -1, 0, \quad 1, \quad 2, \quad 3.$$
$$y = F(x) = -7, \quad 6, \quad 7, 2, -3, -2, 11.$$

Plotting the points $(-3, -7)$, $(-2, 6)$, etc , we obtain the graph of the function $x^3 - 6x + 2$ (see Fig 7).

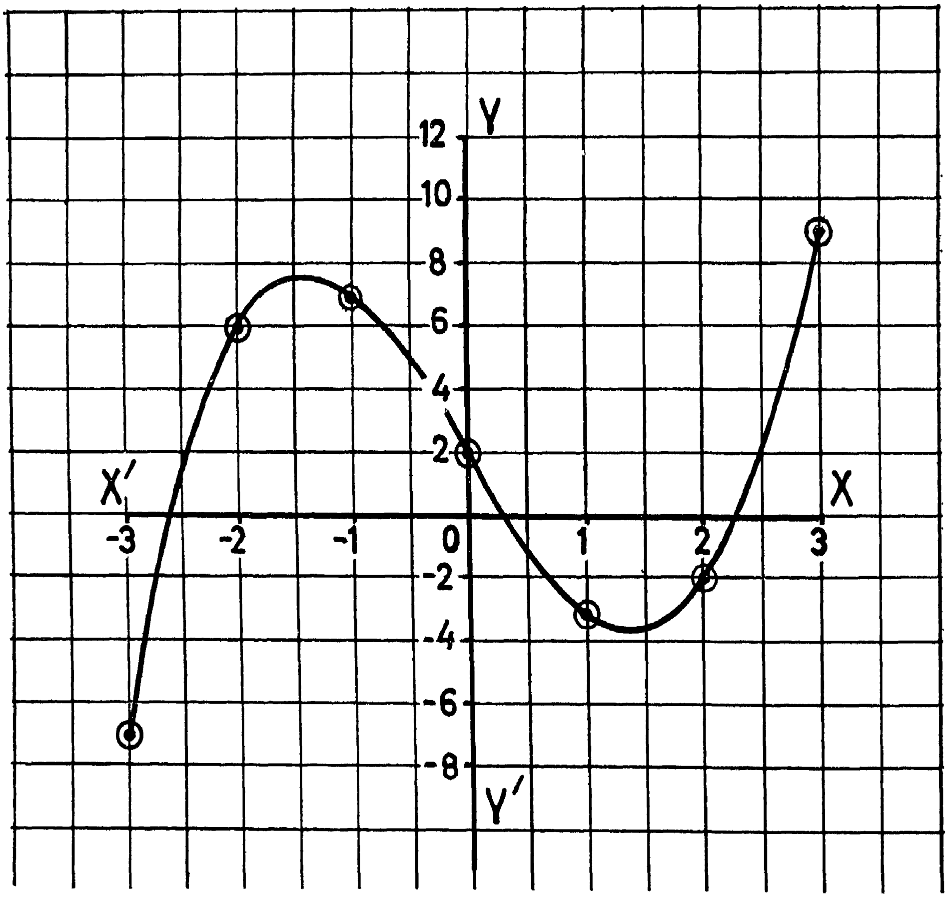


Fig 7

Exercises C

See if you can answer these questions on the graph of $x^3 - 6x + 2$

1. For negative values of x what is the highest value of y on the graph?
2. For positive values of x what is the lowest value of y?
3. For the part of the graph above the x-axis, is the value of y positive or negative?
4. For the part of the graph below the x-axis, is the value of y positive or negative?
5. (*a*) For how many values of x is the value of y zero?
   (*b*) Where in relation to the x-axis or y-axis do these occur?
6. (*a*) For how many values of y is x zero?
   (*b*) Where do these occur?

Dependent and Independent Variables

If $y = f(x)$ as, for example, in $y = x^3 - 6x + 2$, then the value of y depends on the value of x, and y is called the **dependent variable.** Because x may be varied at will it is called the **independent variable.**

Exercises D

1. Given $f(x) = 5x - 3$ find $f(3)$, $f(2)$, $f(1)$, $f(6)$, $f(-1)$, $f(-2)$, $f(-3)$
2. If $F(x) = \frac{2}{3}x + 1$, find $F(6)$, $F(3)$, $F(1)$, $F(0)$, $F(-1)$, $F(-3)$, $F(-6)$
3. If $g(x) = 2x + 9$, find $g(2)$, $g(1\frac{1}{2})$, $g(1)$, $g(\frac{1}{2})$, $g(0)$, $g(-1\frac{1}{2})$, $g(-6)$
4. If $y = 5x - 7$, make a table of values of y corresponding to $x = -1, 0, 1, 2, 3, 4$, respectively
5. If $y = 3x + 1\frac{1}{2}$, make a table of values of y corresponding to $x = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$, respectively.
6. If $f(x) = x^2 - 7x + 2$, find $f(2)$, $f(1)$, $f(0)$, $f(-1)$, $f(-2)$
7. If $F(x) = x^2 + 5x - 6$, find $F(-5)$, $F(-3)$, $F(-1)$, $F(0)$, $F(1)$, $F(3)$, $F(5)$.
8. If $G(x) = x^3 + 2x - 5$, find $G(-4)$, $G(-2)$, $G(0)$, $G(2)$, $G(4)$
9. If $f(x) = x^2 - 2x + 1$, find $f(1\frac{1}{2})$, $f(\frac{1}{2})$, $f(-\frac{1}{2})$, $f(-1\frac{1}{2})$.

Chapter 2

THE LINEAR FUNCTION AND THE LINEAR EQUATION (1)

Example (1)

We see from Chapter 1 that $3x + 1$ is called a **first degree** or **linear function** of x, and that the graph of the **function** $3x + 1$ is the same as the graph of the **equation** $y = 3x + 1$

On the opposite page are the graphs of the functions

$$(a)\ \tfrac{1}{3}x,\ x,\ 4x, \quad \text{and} \quad (b)\ -\tfrac{1}{3}x,\ -x,\ -4x.$$

The following tables of values were used in drawing $y = \frac{1}{3}x$ and $y = -\frac{1}{3}x$, respectively

$y = \frac{1}{3}x$

x	-6	-3	3	6
y	-2	-1	1	2

$y = -\frac{1}{3}x$

x	-6	-3	3	6
y	2	1	-1	-2

Exercises

1. What kind of line describes each of these six graphs?

2. Through which point does each graph pass?

3. Say in which two quadrants the graphs of these functions lie

(*a*) $\frac{1}{3}x$, x, $4x$

(*b*) $-\frac{1}{3}x$, $-x$, $-4x$.

4. Hence say whether the graph slopes upward or downward from left to right

(*a*) when the coefficient of x is positive,

(*b*) when the coefficient of x is negative

5. State how the slope of the graphs in group (*a*) and in group (*b*) varies with the size of the coefficient of x.

6. (*a*) Select any two points for which the y-co-ordinate is four times the x-co-ordinate in each case, for example, $x = -1\frac{1}{2}$, $y = -6$, etc

(*b*) On which graph do these points lie?

7. Select any two other points on this graph Write down their co-ordinates and show that in each case the y-co-ordinate is four times the x-co-ordinate

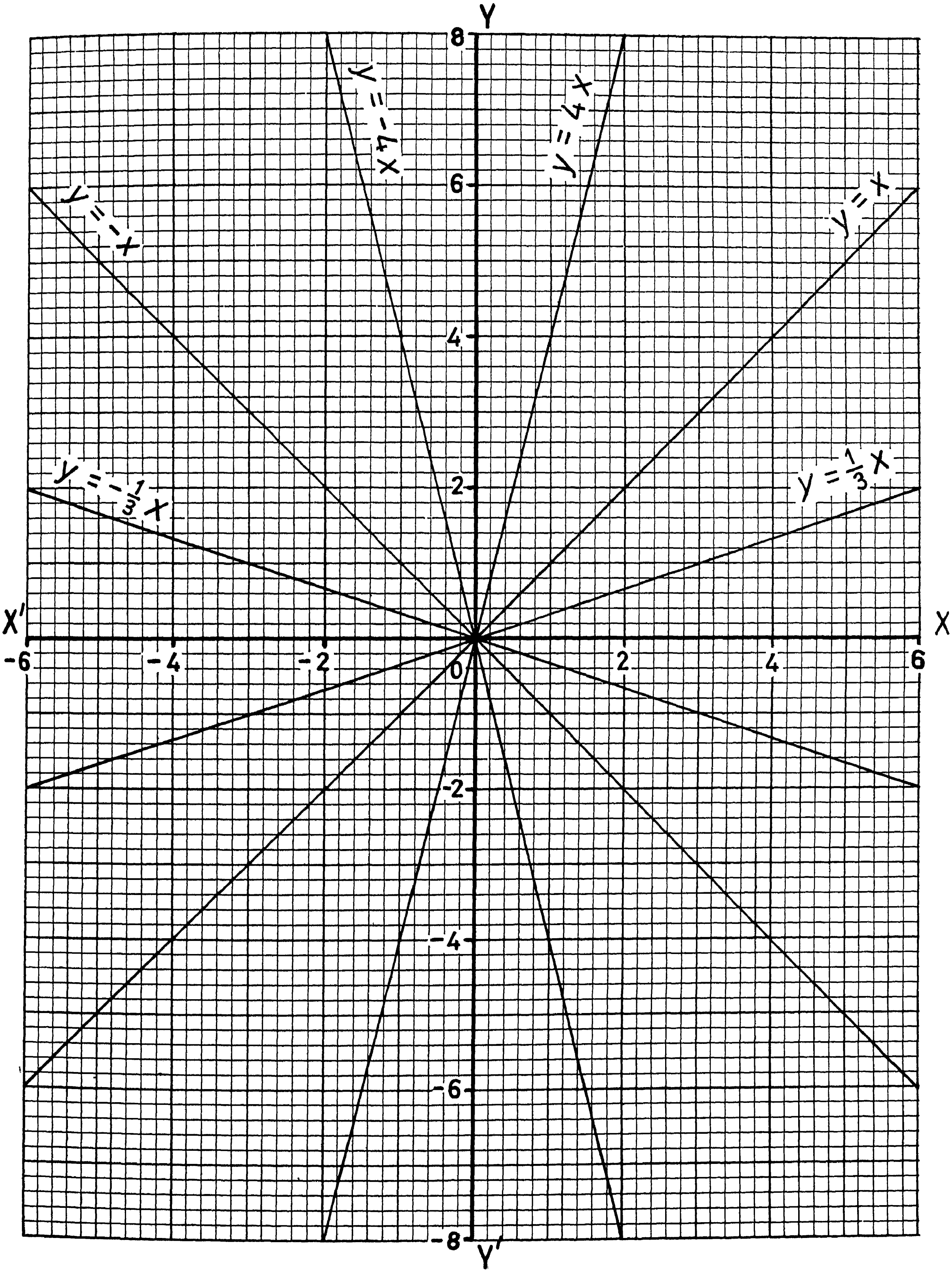
8. On a single graph sheet, and using the same scales, draw the graphs of

$$(\text{i})\ \tfrac{1}{2}x,\ 2x,\ 3x, \quad \text{and} \quad (\text{ii})\ -\tfrac{1}{2}x,\ -2x,\ -3x$$

(You should compile suitable tables as shown in the cases of $\frac{1}{3}x$ and $-\frac{1}{3}x$)

Note The equations of all the graphs in Example (1) are of the form $y = mx$.

GRAPHS OF $\pm\frac{1}{3}x$, $\pm x$, $\pm 4x$



Example (2)

Let us try to discover the equations of the three parallel graphs in the diagram.

Exercises

1. Consider the middle graph whose equation is of the form $y = mx$

(*a*) Complete the following table which gives the co-ordinates of some points on this graph

$$x = -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3$$
$$y = -6,\ .$$

(*b*) Use this table to find the value of m

(*c*) Write down the equation of this graph

2. (*a*) Complete the following table which gives the co-ordinates of some points on the upper graph

$$x = -3,\ -2,\ -1,\ 0,\ 1$$
$$y = .$$

(*b*) Say how each y-co-ordinate could be derived

(i) From the corresponding y-co-ordinate in the previous table

(ii) From the corresponding x-co-ordinate

(*c*) Write down the equation of the upper graph

3. In the same way find the equation of the lower graph

4. State the coefficient of x in the equation of each of these **parallel** graphs

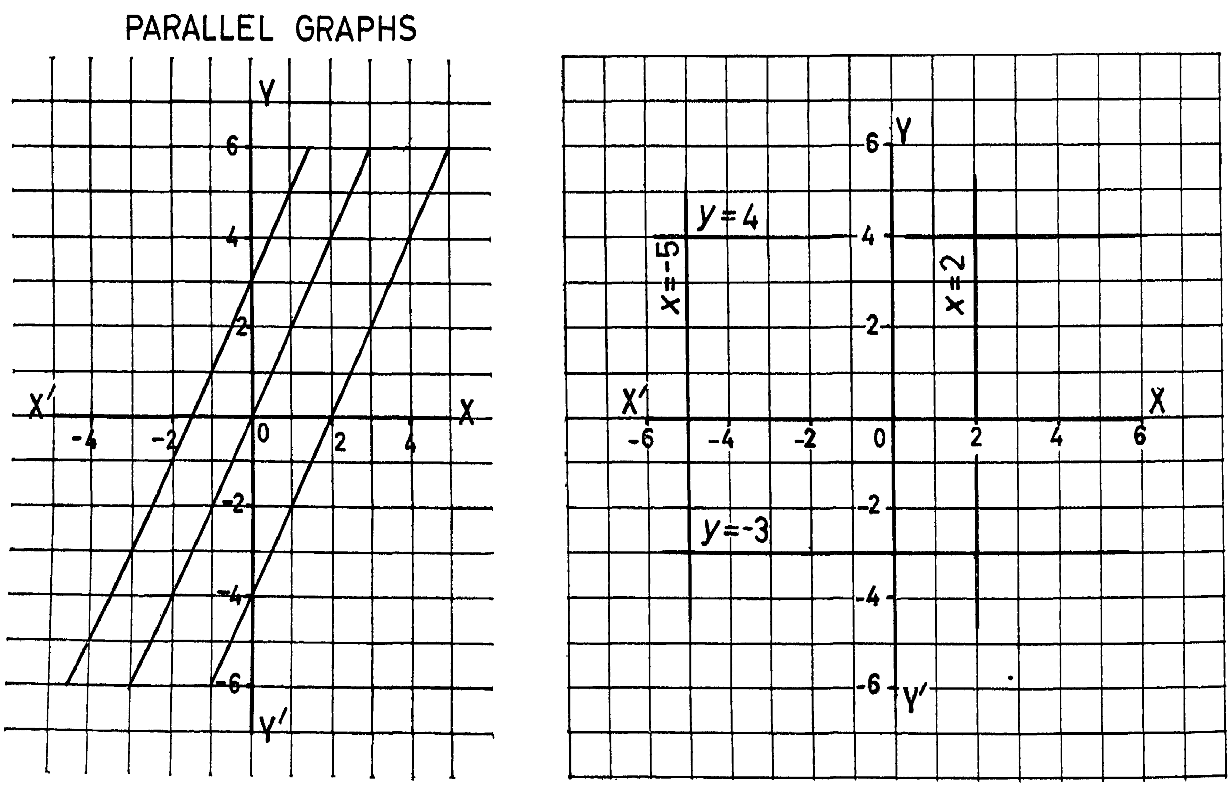


FIG 8

5. (a) Write down the y-co-ordinate* of the point where each graph crosses the y-axis.
(b) Compare this with the constant term in the appropriate equation
6. (a) State which of the following equations give graphs parallel to those in the diagram:

(i) $y = 2x - 1$. (ii) $y = 3x + 2$ (iii) $y = x - 9$
(iv) $y = -2x - 3$ (v) $y = x + 2$. (vi) $y = 2x + 7$

(b) Write down the y-co-ordinates of the points where the graphs of these equations cross the y-axis
7. Write down the equations whose graphs are parallel to those in the diagram and which cross the y-axis where (i) $y = 1$ (ii) $y = -7$ (iii) $y = \frac{3}{4}$
8. Consider the functions (i) $-3x$ (ii) $-3x + 2$ (iii) $-3x - 1$.

(a) State, with reasons, whether or not their graphs are parallel
(b) Write down the y-co-ordinates of the points where these graphs cross the y-axis
(c) Draw the graphs from $x = -6$ to $x = 6$
(d) Consider the equation of the upper graph you have drawn, namely, $y = -3x + 2$

(i) Show from the equation that the following points will lie on the graph $(1, -1)$, $(-1, 5)$, $(2\frac{1}{2}, -5\frac{1}{2})$
(ii) Select any other three points on this graph and show that their co-ordinates satisfy the equation $y = -3x + 2$

Summary of Deductions from Examples (1) and (2)

1 For all numerical values of m the equation $y = mx$ represents a straight line through the origin
2 If m is positive the graph lies in the first and third quadrants sloping upwards from left to right
3 If m is negative the graph lies in the second and fourth quadrants sloping downwards from left to right
4 The coefficient m is a measure of the slope of the graph
5 For all numerical values of m and c the equation $y = mx + c$ represents a straight line parallel to $y = mx$ and cutting off an intercept c from the y-axis
6 As any simple equation in x and y can be reduced to the form $y = mx + c$ (or $y = mx$) its graph will be a straight line The equation is thus called a **linear** equation and the function $mx + c$ is called a **linear** function
7 (a) The equation $y = 4$ indicates that for all values of x the ordinate is constant The graph is thus a line parallel to the x-axis and 4 units above it Similarly, $y = -3$ is a line parallel to the x-axis and 3 units below it
(b) The graphs of $x = 2$, and $x = -5$ are straight lines parallel to the y-axis, 2 units to the right and 5 units to the left of it, respectively (see Fig. 8 opposite)
8. In saying that, for example, $y = 5x - 3$ is the equation of a particular graph two conditions are implied, namely

(a) that all points whose co-ordinates satisfy the equation lie on the graph,
(b) that the co-ordinates of all points which lie on the graph satisfy the equation

* This co-ordinate is called the **intercept** of the graph on the y-axis

Example (3)

To draw the graph whose equation is $3x + 2y = 12$, the pupil should follow the steps given in each of the methods shown below

A *Table of Values Method*

1. First change $3x + 2y = 12$ into the form $y = mx + c$.

$$\text{Say } 2y = \quad .$$
$$y = \ldots$$

2. Now select up to four values of x which will make the corresponding values of y whole numbers, if possible Here x should be an even number. Complete the table by selecting two more values of x and calculating all four values of y

x	-2			10
y				

3. Decide on the scales according to the ranges of values in the table.

Scale: For x 6 inches = units. . 1 inch = . . units

For y if 8 inches is taken to represent 20 units 1 inch = . . units

4. Plot the four points and join them to form the straight line graph

Note Since a straight line can be drawn when any two points on it are known, the table need contain only two pairs of values The others are used as a check

B *The Intercept Method*

1. In the equation $3x + 2y = 12$ putting $x = 0$ gives $y =$., which is the intercept of the graph on the y-axis

2. Similarly, putting $y = 0$ gives $x =$, which is the intercept of the graph on the x-axis

3. Joining these two points gives the graph required

Note 1 Method B is usually the quicker and safer method, but the intercepts may involve fractions of a unit. This would make it difficult to draw the graph accurately Consider

$$7x - 3y = 5$$

If $x = 0$ $\qquad y = -\frac{5}{3}$ (intercept on y-axis)

If $y = 0$ $\qquad x = \frac{5}{7}$ (intercept on x-axis)

In this case two integral values of x and y are found by trial without altering the form of the equation Here $x = 2$ gives $y = 3$, and $x = 5$ gives $y = 10$

The graph can now be drawn by joining the points (2, 3) and (5, 10)

In some cases one of the points chosen may be an intercept.

Note 2 The intercept method cannot be used for equations of the form $y = mx$ which pass through the origin

Scales

As the units measured along each axis have to be spaced out evenly, a scale for x and a scale for y are necessary The scale will depend on (i) the range of values, and (ii) the number of inches available It should normally be as large as possible—that is, there should be as few units to each inch as possible

To make the best use of the ten divisions to each inch, the number of units to 1 inch should be one of the following

(*a*) A *factor* of 10, such as 1, 2, $2\frac{1}{2}$, 5, 10, so that each unit is shown by an exact number of divisions For example, if 5 units = 10 divisions (1 inch), then 1 unit = 2 divisions

(*b*) A *multiple* of 10, such as 20, 30, 40, etc , so that each division represents an exact number of units For example, if 1 inch = 30 units, then $\frac{1}{10}$ inch = 3 units To arrive at the scale figure

(i) divide the range of values by the number of inches on the graph paper,

(ii) choose the factor or multiple of 10 equal to or greater than your quotient

For example

(*a*) If the value of x varies from -4 to 8, giving a range of 12 units to 6 inches of graph paper, 6 inches = 12 units, and 1 inch = 2 units, or 1 unit = $\frac{1}{2}$ inch.

OR

(*b*) If the value of y varies from -56 to 83, the range chosen will normally be from -60 to 90, a total of 150 units, and there are 8 inches of graph paper, so 8 inches = 150 units or 1 inch = $18\frac{3}{4}$ units

Choose 1 inch = 20 units, and then $\frac{1}{10}$ inch = 2 units

Graph paper is normally 8 inches by 6 inches, and in choosing scales for x and y the greater length is normally given to the variable with the greater range of values. For example, if the x range = 21 units and the y range = 157 units

	For x		For y
	6 inches = 21 units		8 inches = 157 units
	1 inch = $3\frac{1}{2}$ units		1 inch = 19+ units
Choose	1 inch = 5 units	Choose	1 inch = 20 units

But if the choice of 6 inches or 8 inches makes a difference to the smaller range and not to the larger, we may choose the greater length for the smaller range

For example, if the x range = 35 units and the y range = 54 units

	For x		For y
	8 inches = 35 units		6 inches = 54 units
	1 inch = 4+ units		1 inch = 9 units
Choose	1 inch = 5 units	Choose	1 inch = 10 units

Note. Had we used 6 inches = 35 units for x this would have given a scale of 1 inch = 10 units

Exercise

Select the most suitable scales for x and y in each of the following cases:

	Range of values		Size of graph sheet
	For x	For y	
1	−3 to 3	2 to 18	8 in × 6 in
2	−4 to 6	−20 to 20	8 in × 6 in
3	3 to 19	−27 to 27	8 in × 6 in
4	0 to 40	0 to 29	9 in × 7 in
5	−4 to 4	5 to 18	9 in × 7 in
6	0 to 55	23 to 171	8 in × 6 in
7	2 to 18	−5 to −35	8 in × 6 in
8	−5 to 5	33 to −4	20 cm × 15 cm
9	40 to 220	56 to 184	20 cm × 15 cm
10	20 to 160	74 to −110	20 cm × 15 cm
11	1 to 11	0 to 20	8 in × 6 in
12	32 to 212	0 to 100	8 in × 6 in

Exercises on Chapter 2

1. By finding four points on each, plot the graphs of the following equations from $x = -3$ to $x = 3$ Use one graph sheet for each set of three.

(*a*) (i) $y = x$. (ii) $y = x + 3$ (iii) $y + 2 = x$
(*b*) (i) $y = -x$. (ii) $y + x = 5$. (iii) $y = -x - 4$
(*c*) (i) $y = 4x$. (ii) $y - 1 = 4x$ (iii) $y + 3 = 4x$.
(*d*) (i) $y = x + 5$. (ii) $y = 5x$ (iii) $y + 5x = 1$.

2. By finding the intercepts on the axes or by joining two suitable points, plot the graphs of the following equations Use one graph sheet for each set of three

(*a*) (i) $x + 2y = 0$ (ii) $x + 2y = 7$ (iii) $x - 2y = 3$.
(*b*) (i) $2x + 3y = 6$ (ii) $2x + 3y = 12$. (iii) $3x - 2y = 6$
(*c*) (i) $5x + 2y = 10$ (ii) $5x - 2y = 10$ (iii) $5x = 2y$
(*d*) (i) $x + 2 = 0$ (ii) $y - 3 = 0$. (iii) $4x + 3y = 0$

3. Draw on the same diagram the graphs whose equations are (i) $x + 3y = 6$, (ii) $2x + y = 7$, (iii) $4x - y = 11$, and show that all three pass through one point Write down the co-ordinates of this point

4. Find the area included between the graphs $x = -5$, $x = 2$, $y = -4$, and $y = 1$

5. Calculate the values of the function $4x - 5$ for values of x which increase by 2, namely, $x = -1, 1, 3, 5, 7$ How do the corresponding values of $4x - 5$ change? Say how this is shown in the graph of $4x - 5$

6. Calculate the values of the function $17 - 2x$ for values of x which increase by 3, namely, $x = -2, 1, 4, 7, 10$. How do the corresponding values of $17 - 2x$ change? Say how this is shown in the graph of $17 - 2x$.

7. Draw the graph of $2x - 3$ using values of $x = -4, 0, 4$. From your graph:

(*a*) Read off values of the function $2x - 3$ when $x = -3, 1, 2, -2\frac{1}{2}$, respectively

(*b*) Read off values of x for which $2x - 3 = -7, -1, 3, -6, 4, 4\frac{1}{2}$, respectively

8. On one graph sheet draw the graphs of $x + 2 = 0$, $y - 3 = 0$, and $2x - 3y = 2$ Use your graphs to calculate the area enclosed by the three graph lines

Chapter 3

THE LINEAR EQUATION (2)

Example (1)

Solution of Simultaneous Equations

Solve graphically the following equations

(i) $y - 2x = 8$ (ii) $4x + 7y = 11$

The pupil should draw the graphs by following out the stages in the exercises and compare the result with the complete graphs shown opposite.

Exercises

1. In the equation $y - 2x = 8$ if $y = 0$, $x =$. , and if $x = 0$, $y =$
2. Thereofre, the intercepts on the x and y axes are and , respectively, and the graph of $y - 2x = 8$ is found by joining the points (, .) and (,)
3. In the equation $4x + 7y = 11$ if $x = 1$, $7y =$ and $y =$, and if $x = 8$, $7y =$ and $y =$ . . .
4. Thus the graph of $4x + 7y = 11$ is found by joining the points (,) and (. ,)
5. The co-ordinates of all points on the first graph satisfy the equation .
6. The co-ordinates of all points on the second graph satisfy the equation
7. The only point which lies on both graphs has co-ordinates (, .)
8. Thus the solution of the given equations is $x =$., $y =$
9. Verify the solution algebraically

Direction of Slope

A *Graphs opposed in slope*

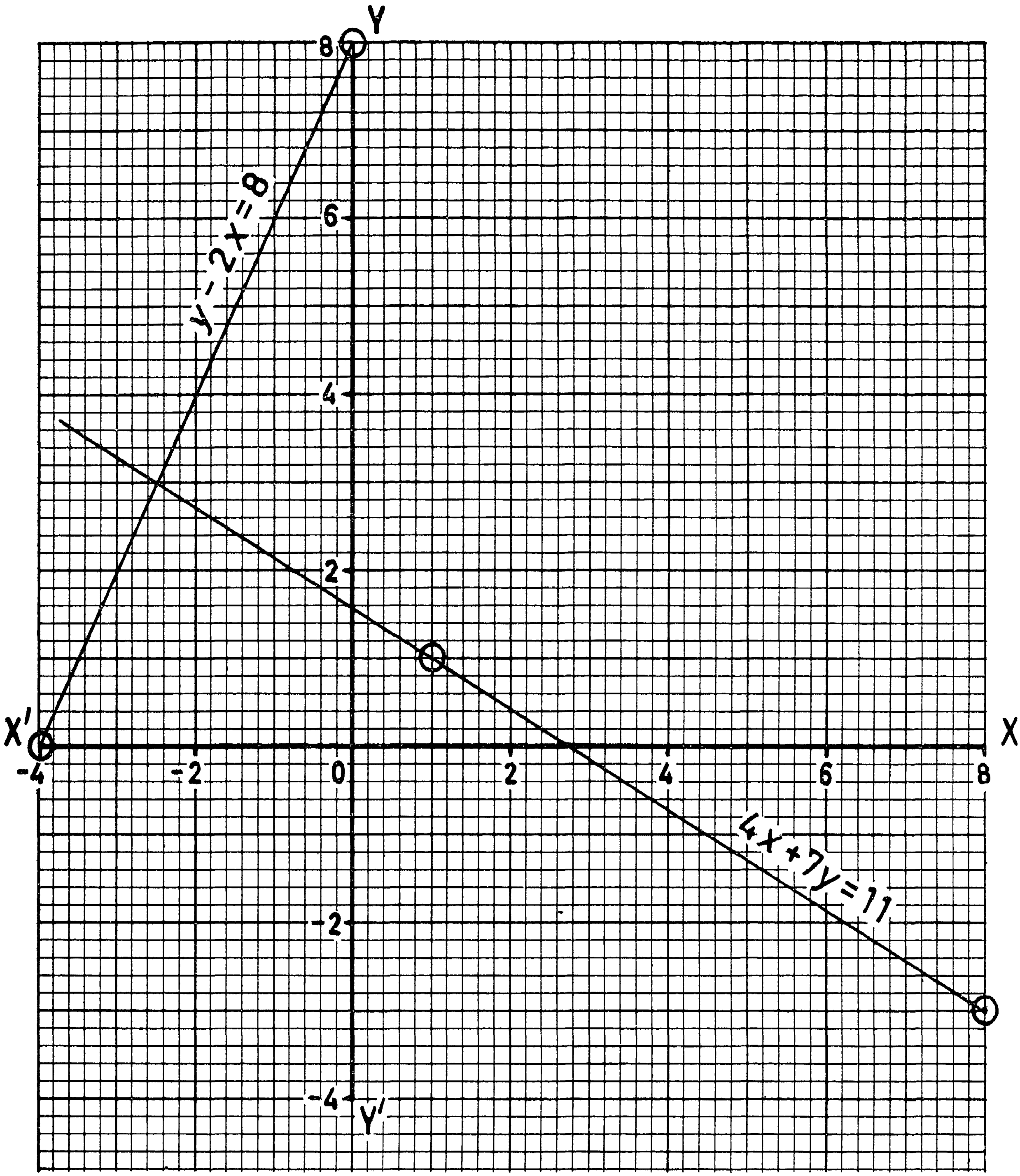
When two graphs are 'opposed' in slope, little difficulty will be found in placing the axes suitably on the page, selecting the scales for x and y, and in obtaining accurately the co-ordinates of their point of intersection Consider, for example, the equations of the graphs drawn in Example (1) and convert these to the form $y = mx + c$ thus

(*a*) If $y - 2x = 8$, $y = 2x + 8$ and the slope of the graph is given by $m = 2$ Thus the graph of $y - 2x = 8$ slopes upwards from left to right

(*b*) If $4x + 7y = 11$ thus $7y = -4x + 11$ and $y = -\frac{4}{7}x + \frac{11}{7}$ The slope of this graph is given by $m = -\frac{4}{7}$ Thus the graph of $4x + 7y = 11$ slopes downwards from left to right

These graphs are opposed in slope Such graphs usually intersect reasonably near the origin and a small error in the direction of either will cause only a small error in their point of intersection The pupil will save time by drawing a rough sketch of the axes and the graphs after calculating two points on each graph.

GRAPHS OF (i) $y - 2x = 8$ (ii) $4x + 7y = 11$



B. *Graphs nearly parallel in slope*

In cases where the direction of slope of the graphs is almost the same, that is, the graphs are nearly parallel, the following points should be observed

(*a*) Accuracy in drawing is essential because a small error in the direction of either graph causes a large shift of the point of intersection

(*b*) The range of values for x and y should be wide.

(*c*) A rough sketch will help to guide you in placing the axes and in choosing the scales for x and y.

C. *Determination of slope*

If a, b, and c are numbers either positive or negative, the equation $ax + by = c$ is called the **general equation of a straight line**

If
$$ax + by = c$$
$$by = -ax + c$$
$$y = -\frac{a}{b}x + \frac{c}{b}$$

Comparing this with $y = mx + k$ we see that the slope of the graph whose equation is $ax + by = c$ is $-\frac{a}{b}$.

Example

(*a*) The slope of $5x + 3y = 11$ is $-\frac{5}{3}$

(*b*) The slope of $3x - 2y = 4$ is $\frac{3}{2}$

(*c*) The slope of $7y = 2x - 3$, that is, of $-2x + 7y = -3$, is $\frac{2}{7}$.

Exercise

Write down the slope of the graph for each of the following equations

(i) $2x + 5y = 7$.	(ii) $3x - 4y = 1$.	(iii) $4x + y = -3$
(iv) $x - 3y = 0$	(v) $\frac{1}{2}x + y = 3$	(vi) $x + \frac{1}{3}y = 6$.
(vii) $2x = 5 - 9y$	(viii) $2y = 2 + 3x$.	(ix) $4x + 4y + 11 = 0$.
(x) $\frac{10x}{3} - \frac{y}{6} = 2$.		

Example (2)

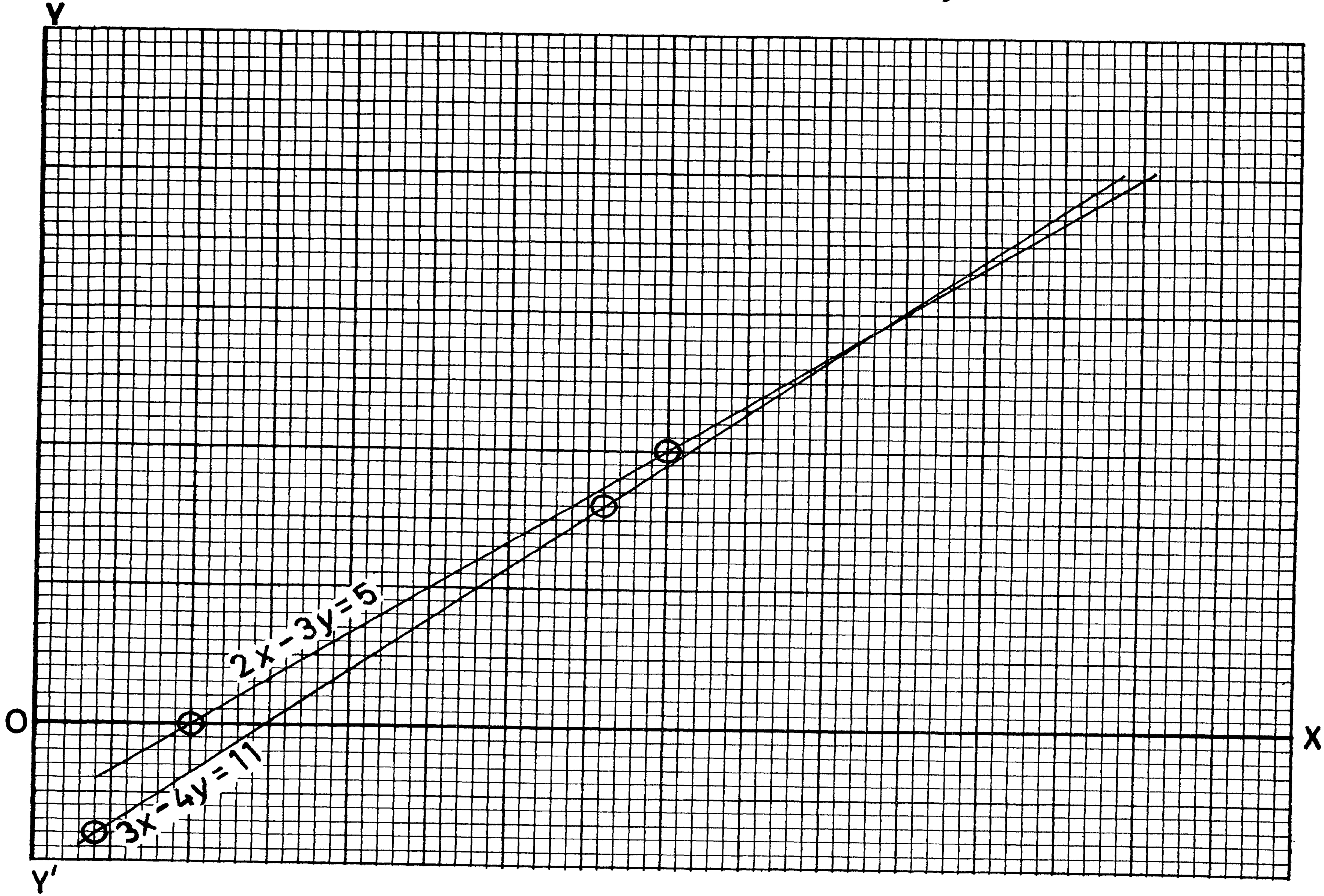
Solution of Simultaneous Equations

Solve graphically the following equations

(i) $2x - 3y = 5$ (ii) $3x - 4y = 11$

The pupil should draw the graphs by following out the stages in the exercises and should compare the results with the completed graphs shown opposite (The scales have been omitted in view of the questions in the exercises)

GRAPHS OF (i) $2x - 3y = 5$ (ii) $3x - 4y = 11$



Exercises

1. In the equation $2x - 3y = 5$ if $y = 0$, $x =$ . ., and if $x = 10, y =$
2. Therefore, the first graph is formed by joining the points (,) and (,).
3. In the equation $3x - 4y = 11$, if $x = 1, y =$, and if $x = 9, y =$
4. Therefore, the second graph is formed by joining the points (,) and (. . , . .).
5. These graphs will be nearly parallel because the slope of $2x - 3y = 5$ is , and of $3x - 4y = 11$ is .
6. Draw two axes each approximately 6 inches long at right angles to each other, crossing in the middle (a spare graph sheet is easiest) Choose, say, 1 division = 5 units for each axis and sketch the graphs
7. From your sketch and choice of values you will see that

   (*a*) The graphs will intersect in the quadrant
   (*b*) At the point of intersection the co-ordinate is greater than the co-ordinate
   (*c*) The x-co-ordinate < 20, so if 8 inches = 20 units the scale for x is 1 inch =
   (*d*) This scale will also suit for y, whose lowest value, calculated in Exercise 3, is . ..

8. Select another pair of values from each equation for accuracy and draw the graphs to these scales
9. The solution shown by the point of intersection of the graphs is $x =$, $y =$ . .
10. Check your answer by substitution of these values in each equation

Note. Graphical solution of simultaneous equations is normally slower and often less accurate than algebraic solution

Example (3)

Equation of a straight line passing through two given points

The graph opposite shows the straight line passing through the points A$(1, -3)$ and B$(3, 2)$ If the equation of this line is $y = mx + c$ the values of m and c will be found by completing the exercises which follow

Exercises

1. Since the point A is on the line its co-ordinates, namely $x = 1, y = -3$, must satisfy the equation $y = mx + c$

$$= \quad m + c \qquad (1)$$

2. Similarly, using the point B (3, 2) we have

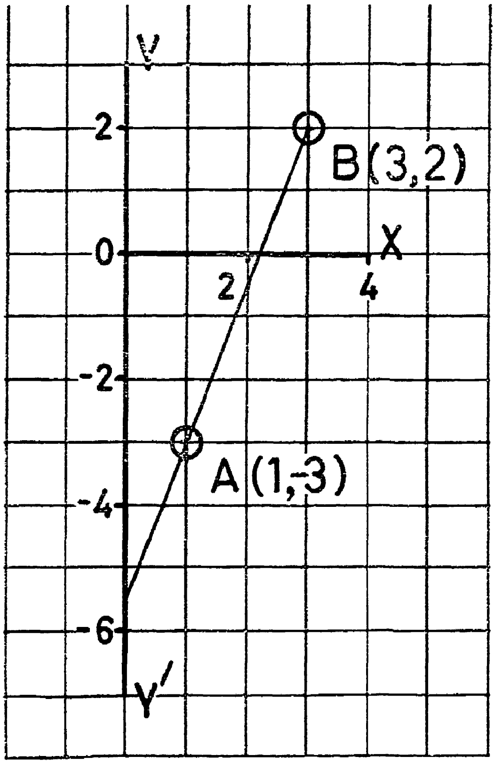
$$.. = \quad m + c \qquad .(2)$$

3. Subtracting equation (2) from equation (1)

$$= \quad m$$

$$\therefore \quad m = . . .$$

4. Substituting in equation (1) $c =$. (Check this value of c with the intercept on the y-axis.)



5. The equation $y = mx + c$ becomes $y = . .$

6. Therefore, simplifying, the equation of the straight line through A(1, —3) and B(3, 2) is

$$=$$

7. Find in the same way the equations of the graphs passing through the following pairs of points

(i) (2, 1) and (4, 5) (ii) (1, 1) and (3, —5) (iii) (2, $5\frac{1}{2}$) and (—1, 1).

(iv) (0, 3) and (—4, 0) (v) (—2, —3) and (4, 5) (vi) (—2, 2) and (3, —6)

Example (4)

Determination of a Linear Law

The method of Example (3) is often useful in dealing with the results of an experiment where the quantities measured appear to be in direct proportion. You will remember that the graph connecting two quantities x and y which are in proportion is a straight line (*Graphs for Interpretation*, Chapter 6) Owing to experimental errors, however, when pairs of values of x and y are plotted the points lie approximately, but not exactly, in a straight line We must draw the line which best appears to show the trend of the plotted points and proceed as for Example (3). The quantities must obey a linear law of the form $y = mx + c$

In an experiment on a certain machine to find the law governing the effort (P lb) required to raise various loads (W lb) the following record was kept

W lb	5	12	21	27	35
P lb	$1\frac{1}{2}$	3	$5\frac{1}{2}$	7	$8\frac{1}{2}$

These pairs of values are shown plotted in the figure opposite, which you must copy before working the exercises

Exercises

1. Turn your ruler on its edge and try to find the line which best shows the trend of the plotted points It may not pass through any of the points but normally will leave some on one side and some on the other
2. Select two points on this line which give as simple values of W and P as possible (These values should not be taken too close together) Here selected points might be where

   (*a*) $W = 3\frac{1}{2}$, $P =$, (*b*) $W = 20$, $P =$.

3. If the law of the machine is $P = aW + b$ proceed as in Example (3) to find a and b
4. Write down the law $P =$ $W +$, and use it to find the effort required on this machine to lift a load of (i) 10 lb, (ii) 14 lb, (iii) 31 lb
   Compare these results with those read directly from the graph
5. Read also from the graph the load which can be lifted by efforts of

   (i) $4\frac{1}{2}$ lb, (ii) 8 lb, (iii) 6 lb 10 oz

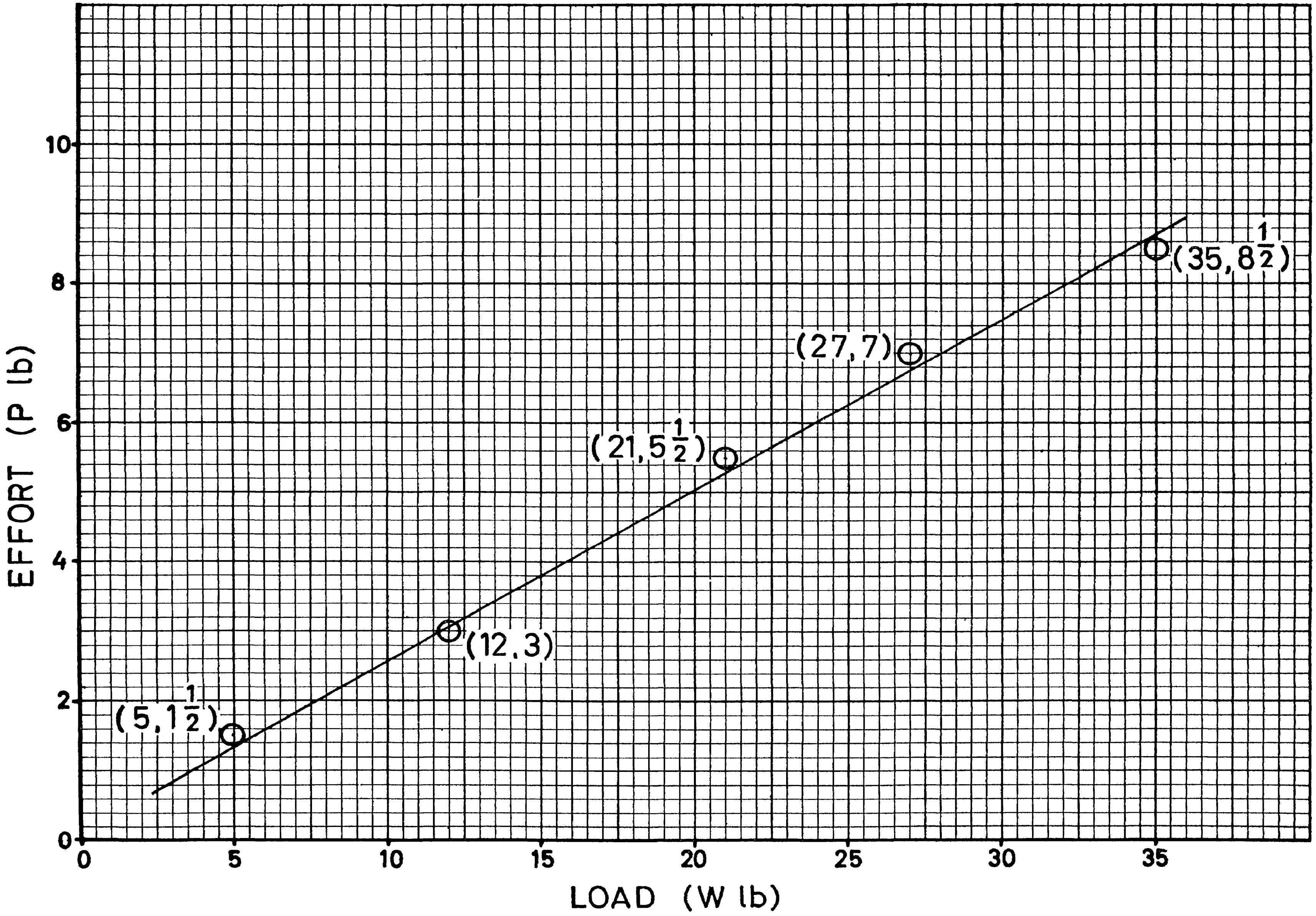
Exercises on Chapter 3

1. Solve graphically the following pairs of simultaneous equations

(i) $y = 2x - 1$	(ii) $y = 3x - 2$	(iii) $3y = 2x$
$y = 11 - 2x$	$x + y = 8$	$x - y = 2\frac{1}{2}$
(iv) $3x + 2y = 1$	(v) $5x - 4y = 1$	(vi) $4x + y - 10 = 0$
$4x - y = 5$	$2x - 3y = 6$	$2x - y = 16$

Check your answers to (i), (iii), and (iv) algebraically, and to (ii), (iv), and (vi) by substitution of the co-ordinates of the point of intersection of the graphs

DETERMINATION OF LINEAR LAW



2. On the same graph sheet draw the graphs of

$$\text{(i) } 3x + 2y = 7, \quad \text{(ii) } 2x - 5y = 11, \quad \text{(iii) } x + 3y = 0,$$

and show that they all meet in one point Find its co-ordinates.

3. (*a*) Using values of x between -6 and 4, on the same graph sheet draw the graphs of

$$\text{(i) } 2x + y = 9, \quad \text{(ii) } x - 2y = -8, \quad \text{(iii) } x + 8y = 12$$

(*b*) Find the co-ordinates of the vertices of the triangle which they form

(*c*) Check by measurement that the graphs of $2x + y = 9$ and $x - 2y = -8$ are at right angles, and, making any necessary measurements, find the area of the triangle

4. The values in the table show the stretched length (L inches) of a spring under various loads (W pounds). Plot these values and draw the straight line which best shows the trend. Use your graph to find the values of a and b in the straight line law $L = a + bW$

W	1	2	3	4	5
L	5 4	5 7	6 1	6 4	6 75

Now from the law find L when

$$\text{(i) } W = 1\tfrac{1}{2}, \quad \text{(ii) } W = 2\tfrac{1}{4}, \quad \text{(iii) } W = 7\tfrac{1}{2}, \quad \text{(iv) } W = 0.$$

Compare these results with those read directly from the graph

Read also from the graph the load which produces a stretched length of

(v) 4 9 inches, (vi) 5 5 inches, (vii) 6 8 inches, (viii) 7 4 inches

5. The following values of the volume (V cu cm) of a certain mass of gas were recorded for different temperatures (t°C) Find graphically a law of the form

$$V = kt + a$$

t	8	14	22	35	45
V	103 3	105 5	107 8	111 2	114 8

Use this law to find V when

$$\text{(i) } t = 20, \quad \text{(ii) } t = 30, \quad \text{(iii) } t = 41$$

Compare these results with those read directly from the graph

Read also from the graph the values of t for which

(iv) $V = 105$ 8, (v) $V = 109$ 5, (vi) $V = 113$ 8

6. Calculate the values of r and s in the straight line law

$$L = rW + s$$

which fit most closely the following records from an experiment on a spiral spring

W (kg)	1	2	3	4	5
L (cm)	22 1	22 55	23 3	23 7	24 1

Use your graph to find L when:

(i) $W = 0$ (explain this result), (ii) $W = 2\ 2$, (iii) $W = 4\ 5$, (iv) $W = 4\ 8$

Compare these results with those calculated from the law for this spring.

7. The table shows experimental values of two quantities z and y which follow the law

$$y = kz^2 + l$$

Plot the values of y against those of z^2 and so find the values of k and l which most closely fit the observations

z	1	2	3	4	5
y	58	68	85	105	136

Note If $x = z^2$ the law becomes $y = kx + l$, which is the equation of a straight line Use the law to calculate y when

(i) $z = 6$, (ii) $z = 10$, (iii) $z = 2\frac{1}{2}$.

8. The table shows the experimental values of the volume (V cu ft) of water in a tank at an interval of time (t sec) after the outflow pipe is opened

t (sec)	0	2	4	6	8
V (cu ft)	596	570	480	305	92

These quantities are connected by the law

$$V = f + gt^2$$

Plot the values of V against those of t^2 and find the values of f and g which best show the trend of the graph Use the law to find V when $t = 1, 3, 5, 7$, respectively.

9. The following table gives the volumes (v cu cm) and corresponding pressures (p cm) of a given mass of gas at a constant temperature

v (cu cm)	50	45	40	35	30	25
p (cm of mercury)	74	81	92 4	106	123 8	147 6

It is known that

$$p = \frac{v}{k} \quad \text{or} \quad p = k \times \frac{1}{v}.$$

Look up the tables of reciprocals for v, that is, values of $\frac{1}{v}$ Plot p against $\frac{1}{v}$ and from the graph find the value of k which best shows the trend

Chapter 4

THE QUADRATIC FUNCTION (I)

Example (1)

$5x^2 + 3x - 4$ is a **quadratic function** of x because the highest power of x it contains is the power 2. The simplest form of quadratic function is the single term x^2

The graph on the opposite page shows how the function x^2 varies as the value of x varies between -6 and $+6$. To draw the graph we let $y = x^2$ and calculate values of y corresponding to various values of x and proceed in the usual way

Exercises

Use the graph to work the following

1. The scale for x is 1 inch = units, $\frac{1}{10}$ inch = unit
2. The scale for y is 1 inch = units, $\frac{1}{10}$ inch = unit
3. Complete the following table to find the squares of numbers

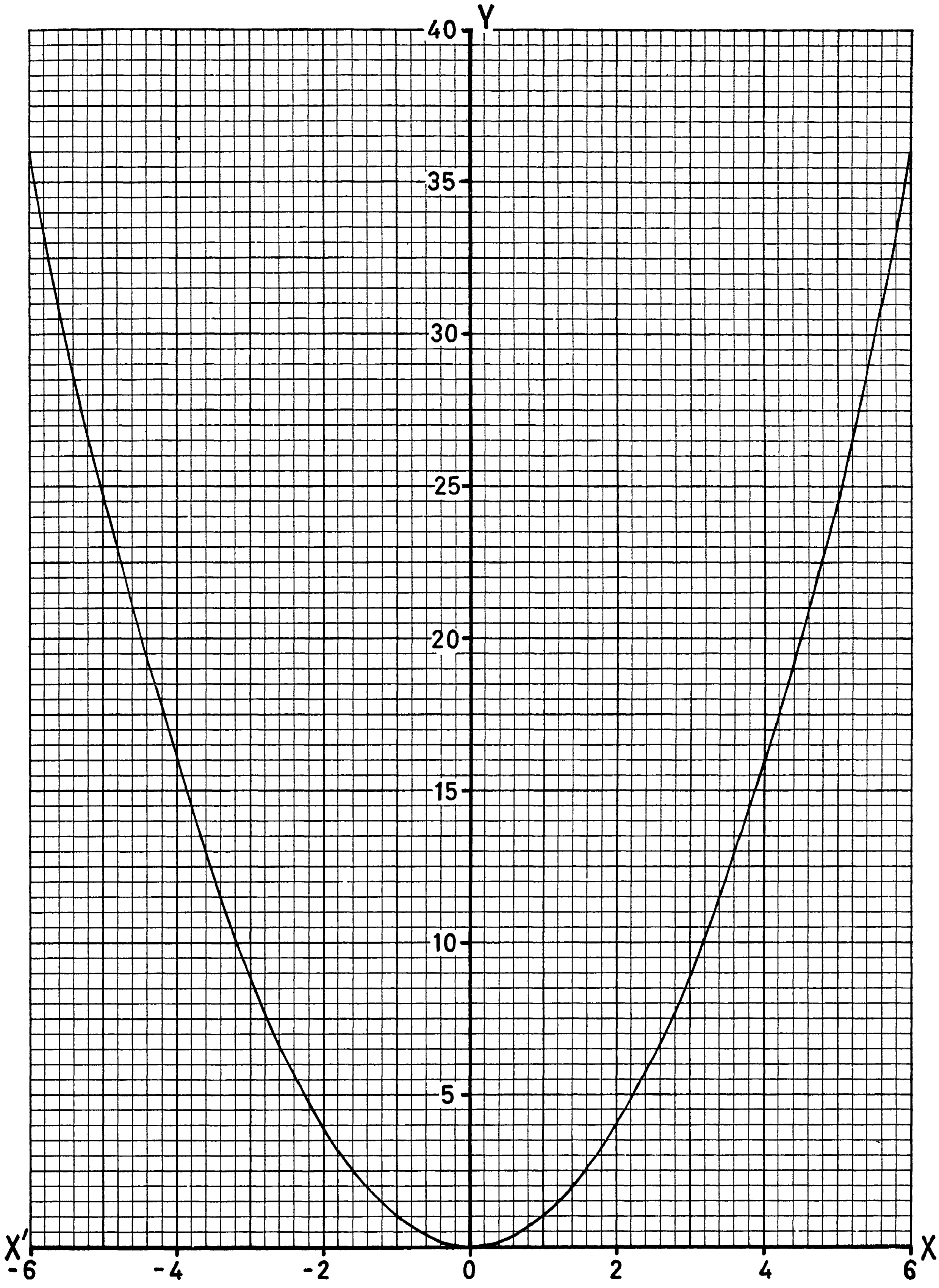
No x	3	5	-2	-6	1 3	-2 7	3 3	-4 6	-5 1	5 7
Square of No $y = x^2$										

4. Complete the following table to find the square roots of numbers

No y	1	16	2	4 4	7 3	9 6	13	17 7	22 1	34 8
Square root of No $x = \pm\sqrt{y}$										

5. Find the areas of squares whose sides are respectively 1 6 in, 2 3 ft, 3 5 cm, 4 6 yd, 5 3 miles
6. Find the lengths of the sides of squares whose areas are respectively 12 25 sq in, 5 75 sq cm, 16 8 sq miles, 30 25 sq yd, 33 7 sq ft
7. The lowest value of y ($= x^2$) is $y =$ Which line is a tangent to (that is, touches) the graph?
8. Write down the co-ordinates of the lowest point on the graph
9. Say for each other value of x^2
   (*a*) How many values of x there are
   (*b*) How these values compare in (i) size, and (ii) sign
10. About which line, therefore, is the graph symmetrical? (This line is called the **axis of symmetry.**)
11. How does the fact that x^2 is always positive affect the position of the graph?
12. (*a*) Write down the increases in x^2 corresponding to unit increases in x from 0 to 1, 1 to 2, 2 to 3, 3 to 4, 4 to 5, 5 to 6
    (*b*) Say how these increases affect the slope of the graph as x increases from 0 to 6

GRAPH OF x^2



Example (2)

The diagram below shows the graphs of the following functions

(i) $(x + 3)^2$ (ii) x^2 (iii) $(x - 2)^2$

Exercises

1. Say how the graphs compare for (i) shape,* and (ii) minimum (lowest) value

2. (*a*) The constant term inside the bracket of the function $(x + 3)^2$ is and the axis of symmetry for its graph is $x =$

(*b*) The constant term inside the bracket of the function $(x - 2)^2$ is and the axis of symmetry for its graph is $x =$

3. Write down the axis of symmetry for the graphs of each of the following functions

(i) $(x + 5)^2$ (ii) $(x + 1)^2$ (iii) $(x - 4)^2$ (iv) $(x - 1\cdot5)^2$

4. Expand (i) $(x + 3)^2$, and (ii) $(x - 2)^2$

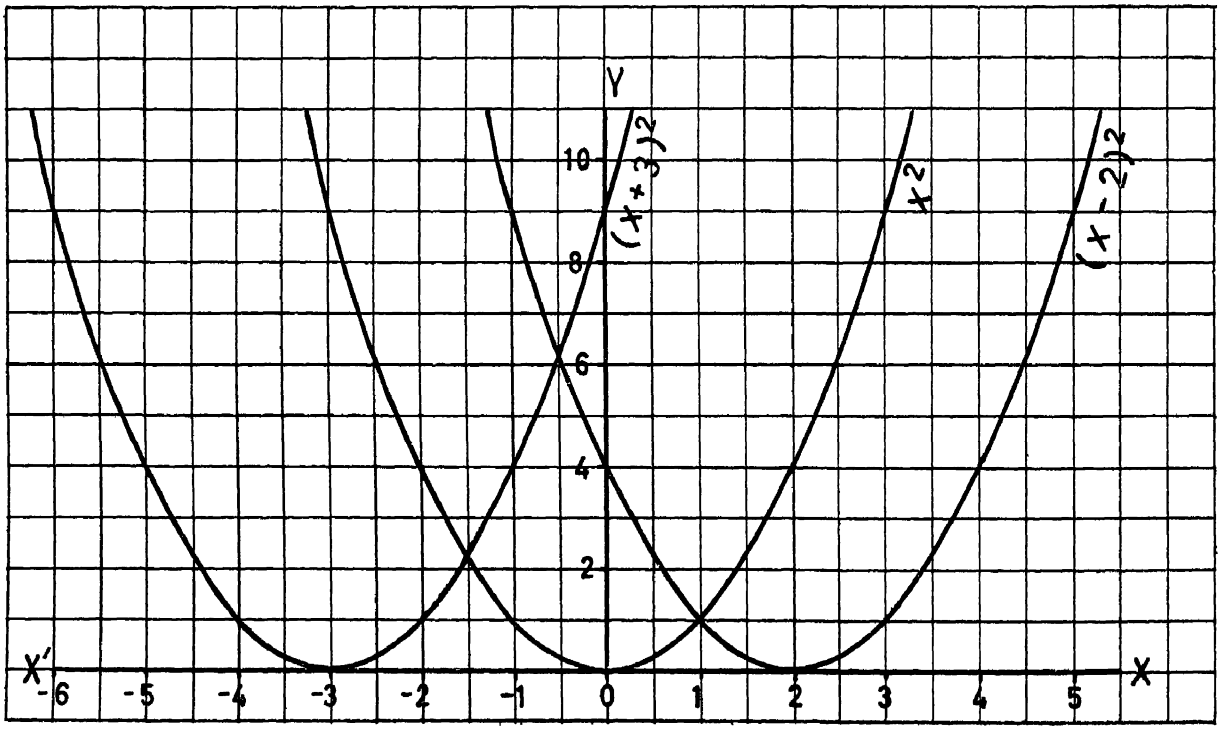
5. (*a*) In the expanded form of $(x + 3)^2$ the constant term is , and its graph cuts the y-axis where $y =$ †

(*b*) In the expanded form of $(x - 2)^2$ the constant term is , and its graph cuts the y-axis where $y =$ †

* The shape of the graph of a quadratic function is called a **parabola**

† The distance from the origin at which a graph cuts the y-axis is called the **intercept** of the graph on the y-axis

GRAPHS OF FUNCTIONS $(x+3)^2, x^2, (x-2)^2$



6. Write down the y-co-ordinates of the points in which the graphs of these functions cut the y-axis

(i) $x^2 - 6x + 9$ (ii) $x^2 + 10x + 25$

(iii) $(x - 4)^2$ (iv) $(x - 7)^2$

7. The function x^2 has no constant term. Therefore, the graph cuts the y-axis where $y =$

Example (3)

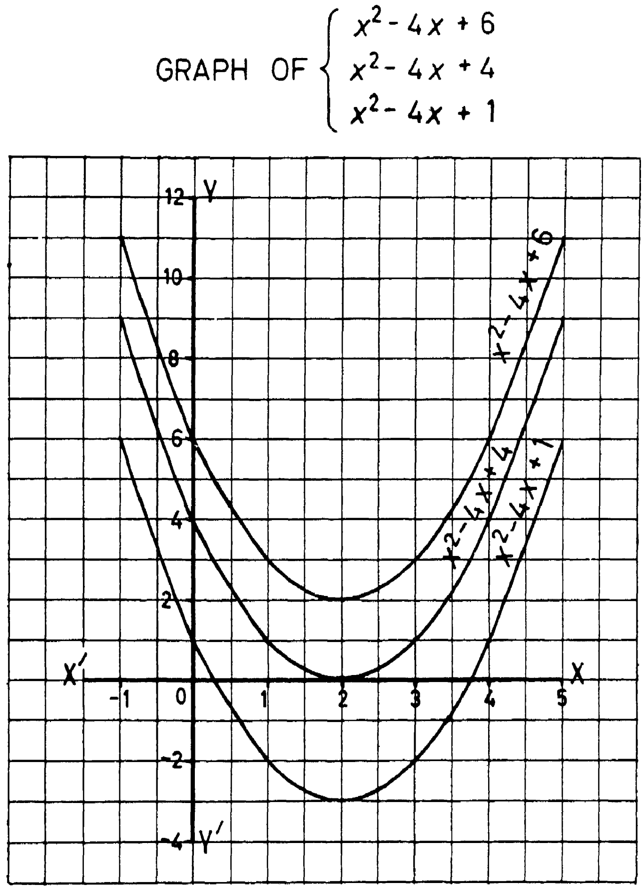
The diagram below shows the graphs of the following functions

$$f(x) = x^2 - 4x + 4 = (x - 2)^2$$
$$F(x) = x^2 - 4x + 6$$
$$G(x) = x^2 - 4x + 1$$

Exercises

1. For each value of x, $F(x)$ is units more than $f(x)$ The graph of $F(x)$ may be derived from the graph of $f(x)$ by moving it units upwards

2. Explain similarly how to derive the graph of $G(x)$ from the graph of $f(x)$.



3. From the graph of $f(x)$ say how you would derive the graph of. (i) $x^2 - 4x + 9$, and (ii) $x^2 - 4x - 2$

4. Name the axis of symmetry for all of these graphs

5. How might this axis of symmetry be obtained from the coefficient of the x term?

6. Write down the axis of symmetry for the graphs of each of the following functions

(i) $x^2 - 6x + 5$	(ii) $x^2 - 6x + 2$.	(iii) $x^2 + 6x + 5$
(iv) $x^2 + 4x + 7$.	(v) $x^2 - 3x - 1$	(vi) $x^2 + 7x - 4$.

7. State the intercepts made by the graphs of $f(x)$, $F(x)$, and $G(x)$ on the y-axis

8. Write down the intercept on the y-axis of each of the graphs of the functions in Exercise **6.**

9. The lowest value of each of the functions $f(x)$, $F(x)$, and $G(x)$ occurs when $x =$ (on the axis of symmetry)

10. Check by working out $f(x)$, $F(x)$, and $G(x)$ for this value of x and compare your answers with the lowest values obtained from the graphs

Example

For the graph of the function $x^2 - 6x - 4$

1 State the equation of the axis of symmetry

2. Find the lowest value of the function (that is, the y-co-ordinate of the lowest point on the graph).

3. Find the intercept made by the graph on the y-axis.

Answer

(1) The axis of symmetry is obtained from $\frac{1}{2}$ the coefficient of the x term with the sign reversed.* Thus it is $x = +3$.

(2) The lowest value of the function is $f(+3)$.

$$f(+3) = 3^2 - 6 . 3 - 4 = -13$$

(3) The intercept on the y-axis occurs where $x = 0$ and so is $f(0) = -4$

Exercises on Chapter 4

1. For the graphs of each of the following functions of x, write down

(*a*) The equation of the axis of symmetry

(*b*) The y-co-ordinate of the lowest point.

(*c*) The intercept on the y-axis

(i) $x^2 - 4x + 5$.	(ii) $x^2 - 8x - 2$
(iii) $x^2 - 5x - 7$.	(iv) $x^2 + x - 4$
(v) $x^2 + 6x + 3$	(vi) $x^2 - 2x + 4$
(vii) $x^2 - x + 5$	(viii) $x^2 - 3\ \ 4x + 5\ \ 8.$
(ix) $x^2 - \frac{4}{3}x + \frac{1}{2}$	(x) $x^2 + \frac{3}{4}x - 3$

* This is true only if the coefficient of x^2 is 1 For the graph of the general quadratic function $ax^2 + bx + c$ the axis symmetry is $-\frac{b}{2a}$ For example, in $3x^2 - 7x + 5$ it is $+\frac{7}{6}$

2. Draw the graph of $(x - 1)^2$ from $x = -2$ to $x = 4$ by writing $y = (x - 1)^2$ and completing the following table

x	-2	-1	0	1	2	3	4
y	9						

Scale for x. 1 inch = 2 units Scale for y: 1 inch = 5 units. On the same graph sheet draw the graphs of $(x + 2)^2$ and $(x - 3)^2$ using the graph already drawn, your knowledge of symmetry, and the lowest points on these graphs.

3. Draw the graph of $(x + 2)^2$ from $x = -5$ to $x = +1$ using the same scales as in the previous question Write $(x + 2)^2$ in expanded form.

On the same graph sheet draw the graphs of $x^2 + 4x + 11$ and $x^2 + 4x + 1$ from your knowledge of the effect on the graph of $(x + 2)^2$ of altering the constant term.

4. Draw the graph of $y = (x - 3)^2$ from $x = 0$ to $x = +6$ By altering the sign of each y-co-ordinate, draw the graph of $y = -(x - 3)^2$.

Write: (i) $(x - 3)^2$ and (ii) $-(x - 3)^2$ in expanded form

Explain the effect on the graph of making the coefficient of x^2 negative

5. On the same graph sheet draw the graphs of x^2, $(x + 1\cdot5)^2$, and $(x - 2\cdot5)^2$ as in Question **2.** For each graph write down

(i) The axis of symmetry.
(ii) The minimum value of y.
(iii) The intercept on the y-axis

6. On the same graph sheet draw the graphs of $(x + 1)^2$, $x^2 + 2x + 7\cdot5$, and $x^2 + 2x - 1\cdot25$, as in Question **3.**

From each graph write down

(i) The axis of symmetry
(ii) The minimum value of y
(iii) The intercept on the y-axis.

7. Write down the axis of symmetry for the graphs of each of the following functions of x

(i) $2x^2 - 5x + 3$	(ii) $3x^2 + 6x - 7$
(iii) $5x^2 + 2x - 1$	(iv) $4x^2 - 7x$.
(v) $-2x^2 - 3x + 4$	(vi) $-6x^2 + x - 3$
(vii) $2x^2 - 6x + 1$.	(viii) $3x^2 + 12x - 5$.

Chapter 5

THE QUADRATIC FUNCTION (2)

Example (1)

To draw the graph of $x^2 - 2x - 3$ shown opposite, we let $y = x^2 - 2x - 3$ and choose values of x equally spaced on either side of the axis of symmetry (namely, $x = +1$) This gives the following table

x	-3	-2	-1	0	1	2	3	4	5
x^2 $-2x$ -3	9 6 -3	4 4 -3	1 2 -3	0 0 -3	1 -2 -3	4 -4 -3	9 -6 -3	16 -8 -3	25 -10 -3
$y = x^2 - 2x - 3$	12	5	0	-3	-4	-3	0	5	12

Note The values of y in this case could also have been calculated from $(x - 1)^2$ since $x^2 - 2x - 3 = (x - 1)^2 - 4$ For example, if $x = -2, y = (-3)^2 - 4 = 5$

Exercises

Use the graphs to complete the following

1. The scale for x is 1 inch = units, 0 1 inch = . unit

2. The scale for y is 1 inch = units, 0 1 inch = . . unit.

3. For the range of values shown, the value of $x^2 - 2x - 3$ $(=y)$ decreases between $x = -$ to $x = +$

4. The lowest value of $x^2 - 2x - 3$ is , and is found where $x =$

5. For the range of values shown, the value of $x^2 - 2x - 3$ increases between $x =$. and $x =$ . .

6. The function is negative for all values of x between $x =$ and $x =$

7. The function is positive for all values of $x >$ (greater than) $x =$ and for all values of $x <$ (less than) $x =$

8. The graph crosses the x-axis where $x =$ and $x =$

9. The graph crosses the y-axis where $y =$.

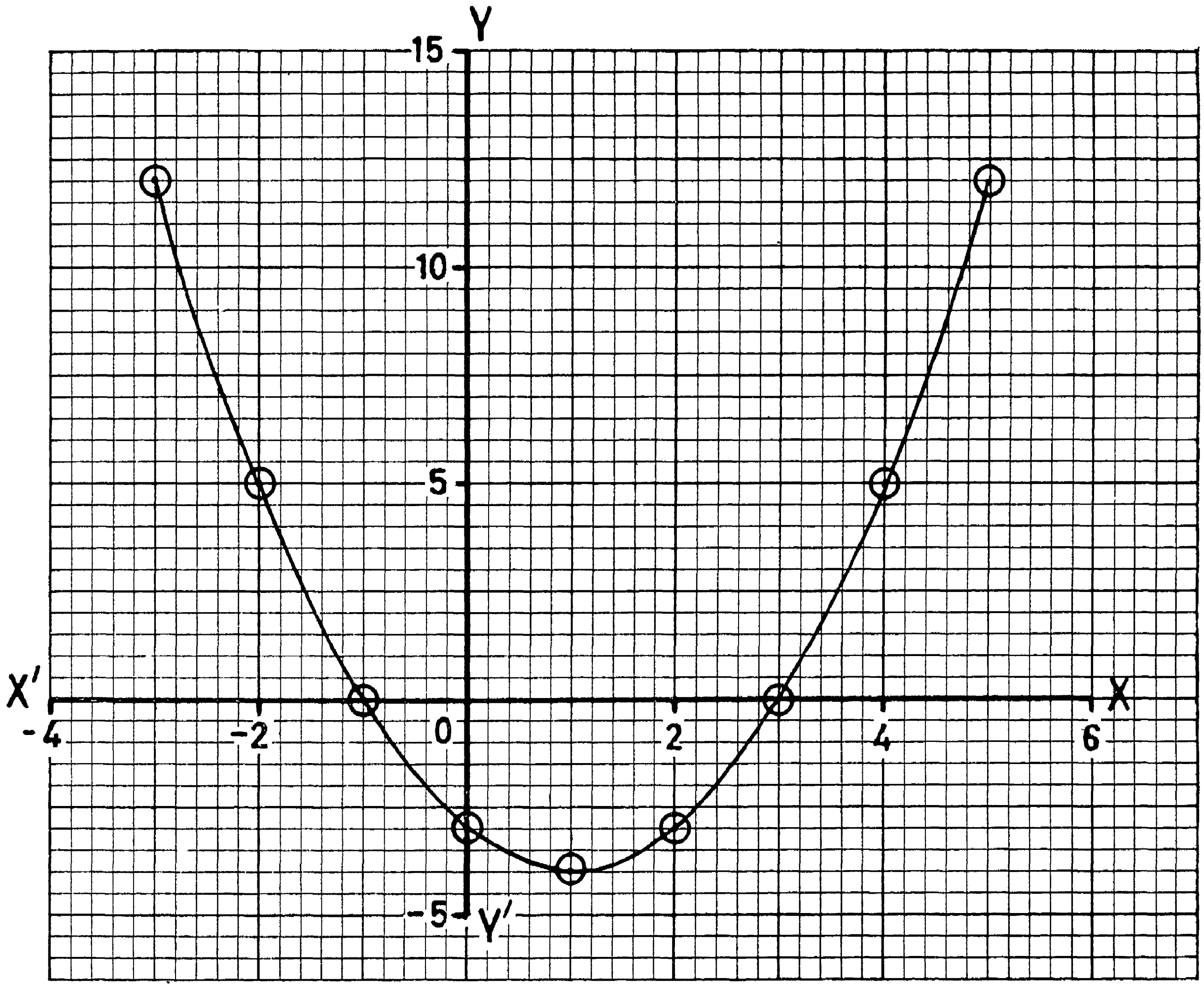
10. Use the graph to fill in the following table

x	$-2\ 5$	$-1\ 8$	$-0\ 9$	$0\ 4$	$1\ 8$	$2\ 8$	$4\ 1$
$y = x^2 - 2x - 3$							

11. Use the graph to fill in both values of x in each part of the following table

$y = x^2 - 2x - 3$	9	$6\ 5$	$3\ 25$	$0\ 75$	$-1\ 5$	$-3\ 25$
x						

GRAPH OF $x^2 - 2x - 3$



Example (2)

The table of values chosen in drawing the graph of $-x^2 + 3x - 2$ is as follows

x	-2	-1	0	1	2	3	4	5
$-x^2$ $+3x$ -2	-4 -6 -2	-1 -3 -2	0 0 -2	-1 3 -2	-4 6 -2	-9 9 -2	-16 12 -2	-25 15 -2
$y = -x^2 + 3x - 2$	-12	-6	-2	0	0	-2	-6	-12

Exercises

Use the graph opposite to complete the following

1. For the range of values shown the value of $-x^2 + 3x - 2$ $(=y)$ increases from $x = -$ . . to $x = +$
2. The highest (or maximum) value of $-x^2 + 3x - 2$ is , and occurs when $x =$
3. For the range of values shown the value of $-x^2 + 3x - 2$ decreases from $x =$ to $x =$
4. The function is positive for values of x between $x =$ and $x =$
5. Because $-x^2$ is always negative, explain the effect of this term on the graph for values of $x < -2$ or > 5
6. The graph crosses the y-axis where $y =$ . .
7. The graph crosses the x-axis where $x =$ and $x =$
8. The axis of symmetry is $x =$

Example (3)

Part of the table of values chosen in drawing the graph of the function $2x^2 + 5x - 10$ is as follows

x	-5	-4	-3	-2	-1	0	1	2
$y = 2x^2 + 5x - 10$	15	2	-7	-12	-13	-10	-3	8

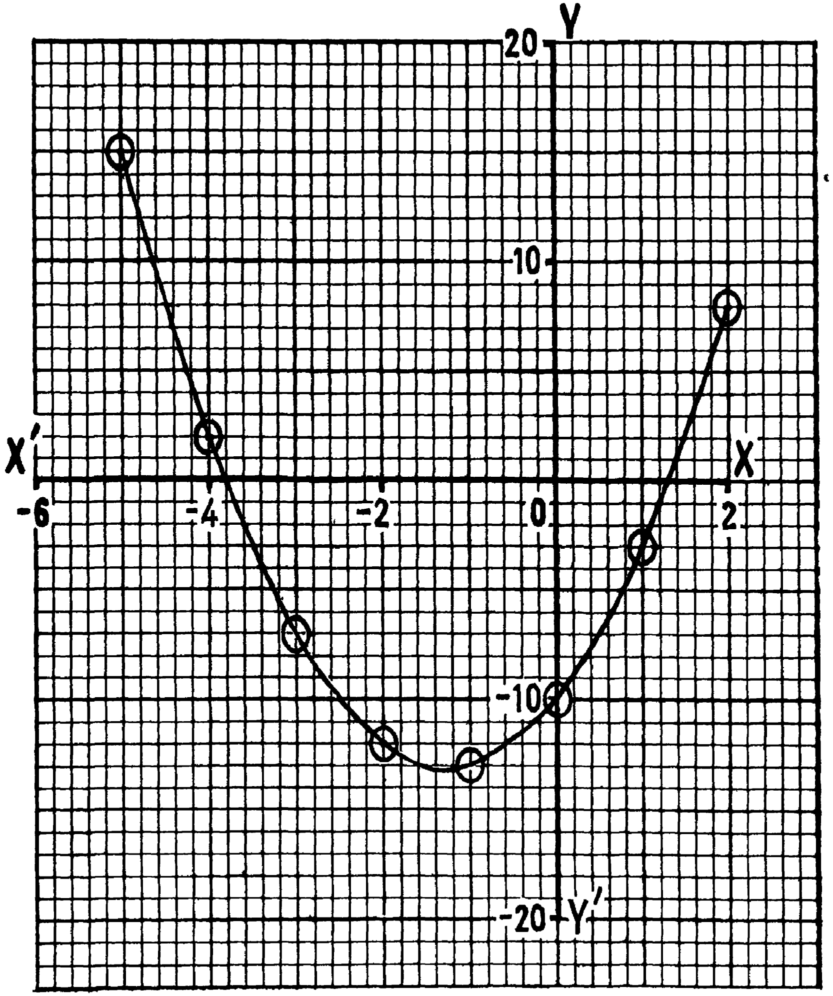
Exercises

Use the graph opposite to work the following.

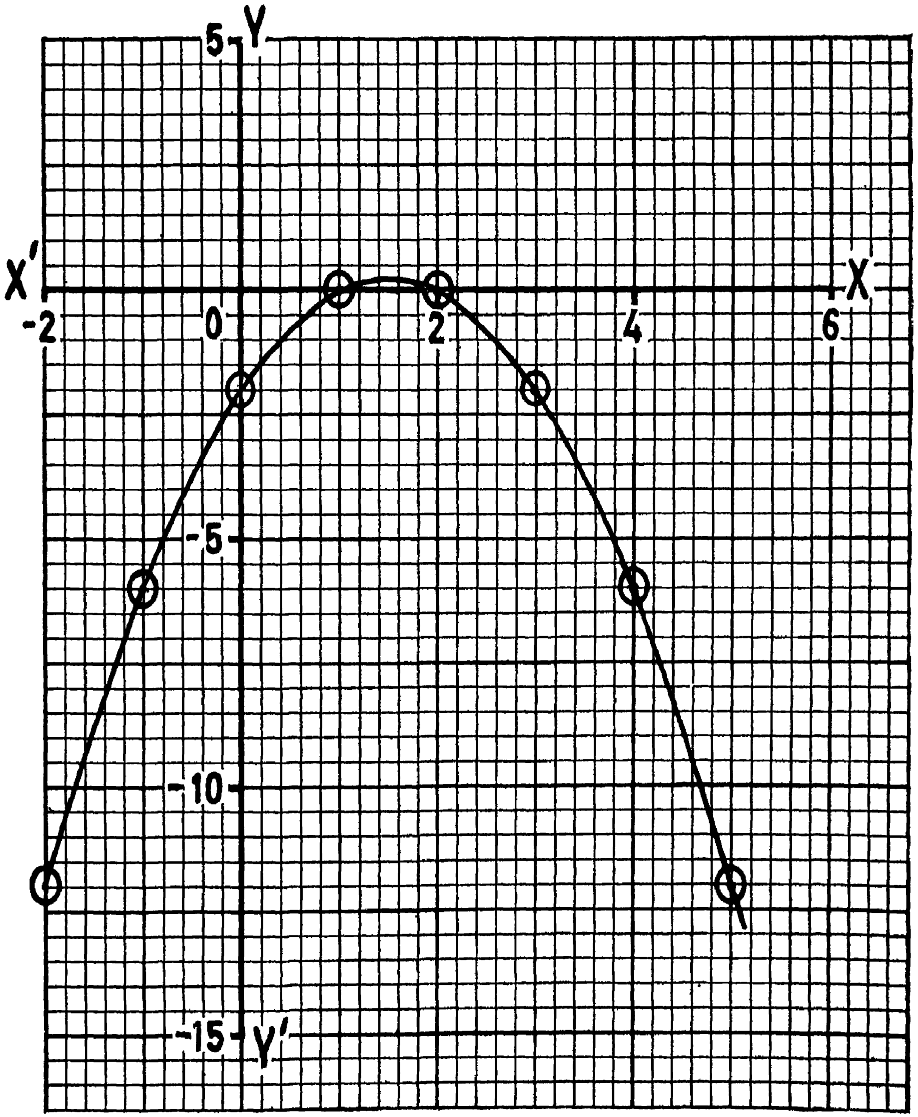
1. The function $2x^2 + 5x - 10$ is negative for all values of x between $x = -$. and $x = +$.
2. The minimum value of the function is (approx) . . , and occurs where $x =$
3. Check your answer to Exercise **2** by calculating the axis of symmetry $\left(-\dfrac{b}{2a}\right)$
4. Find the value of the function corresponding to each of the following values of x

   (i) $x = -4\ 6$ (ii) $x = -2\ 9.$ (iii) $x = -0\ 65$ (iv) $x = 1\ 5$

GRAPH OF $2x^2 + 5x - 10$



GRAPH OF $-x^2 + 3x - 2$



5. Wnte down both values of x correspondmg to each of the followmg values of the function (represented by y)

(1) $y = 5$ (11) $y = -3\ 7$ (111) $y = -8\ 5$

6. (*a*) For which value of y 1s there only one value of x?
(*b*) Where does this occur on the graph?

Tables of Values and Symmetry

Compare the followmg parts of the tables of values for the graphs m Examples (1), (2), and (3)

Example (1) $y = x^2 - 2x - 3$

x	0	1	2
y	-3	-4	-3

Example (2) $y = -x^2 + 3x - 2$

x	0	1	2	3
y	-2	0	0	-2

Example (3) $y = 2x^2 + 5x - 10$

x	-2	-1	0
y	-12	-13	-10

Example (1) The two values of $y = -3$ are symmetrically placed about the value $y = -4$ the lowest value m the table Thus $y = -4$ 1s the mmimum value of y and correspondmgly $x = 1$ 1s the axis of symmetry

Example (2) The two values $y = 0$ and agam the two values $y = -2$ are symmetrically placed about a value midway between these pairs of values So the maximum value of y will be greater than $y = 0$ and will correspond to the value $x = 1\ 5$

Example (3) The values of y m the table are not symmetrical values, and so the lowest value shown, namely, $y = -13$ 1s not the mmimum value The mmimum value will be less than -13 and will be nearer to the line $x = -1$ than to the line $x = -2$

Example (4)

The figure below shows the graph of the function $x^2 - 3x - 1$ between $x = -2$ and $x = 5$

Exercises

Use the graph to work the following

1. Write down the minimum value of $x^2 - 3x - 1$

2. Write down the value of x for which $x^2 - 3x - 1$ is a minimum

3. The value of the function decreases from $x = -2$ to $x =$

4. The value of the function increases for all values of $x >$

5. Say between which two values of x the function is negative

6. State the two ranges of values of x for which the value of the function lies between 2.5 and 5

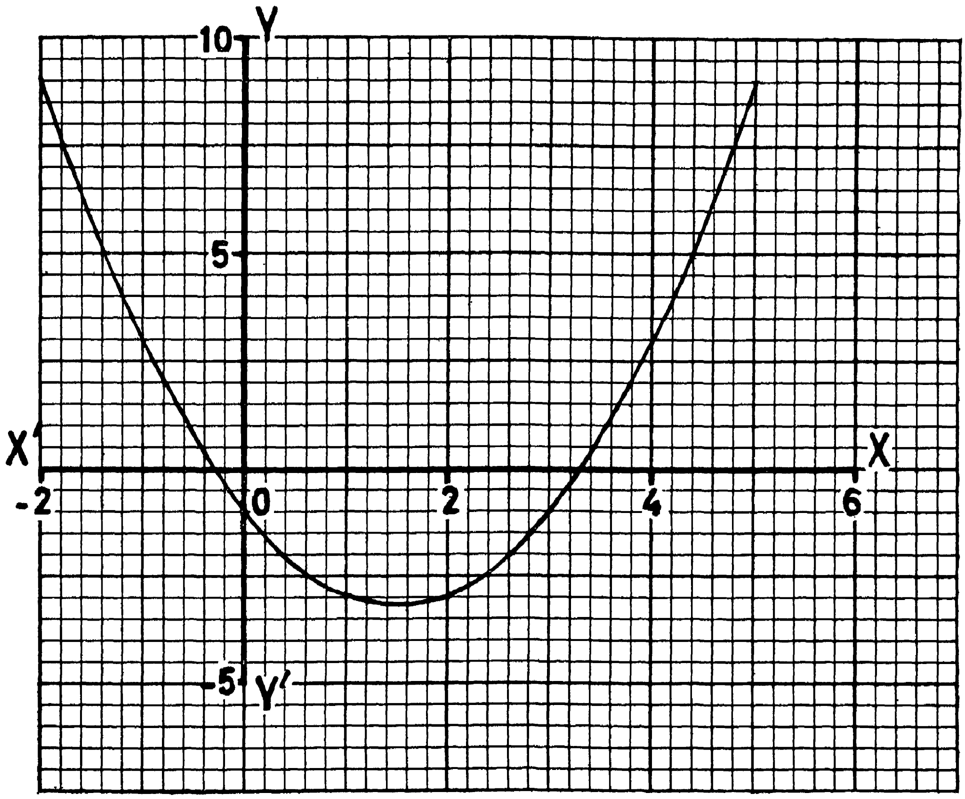
7. Write down the two values of x for which $x^2 - 3x - 1$ has each of the following values

(i) 7 (ii) 1.5 (iii) -0.75 (iv) -2

8. Write down the values of $x^2 - 3x - 1$ for each of the following values of x:

(i) -1.5 (ii) 0.5 (iii) 2.6 (iv) 4.7

GRAPH OF $x^2 - 3x - 1$



Example (5)

The figure opposite shows the graph of the function $5 + 2x - 4x^2$ between $x = -2$ and $x = +3$.

Exercises

Use the graph to work the following:

1. Write down the lowest value *shown* of $5 + 2x - 4x^2$ (i) when x is negative, (ii) when x is positive

2. Describe the behaviour of $5 + 2x - 4x^2$ as x increases (i) from -2 to 0, (ii) from $\frac{1}{2}$ to 3

3. Write down the maximum value of $5 + 2x - 4x^2$

4. Write down the value of x for which $5 + 2x - 4x^2$ is a maximum

5. Read off the value of $5 + 2x - 4x^2$ for each of the following values of x

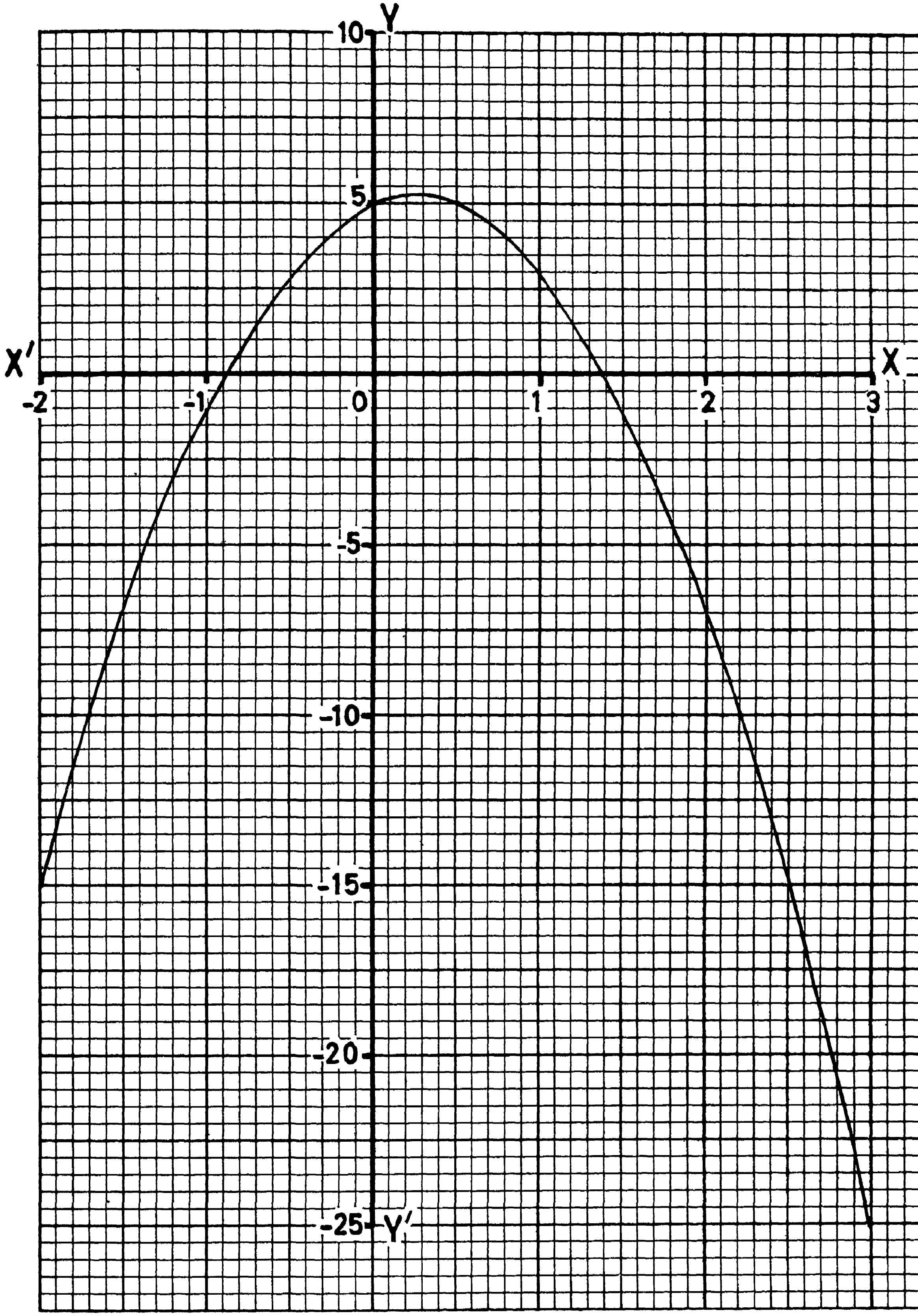
(i) -1.5 (ii) -0.8. (iii) -0.1.
(iv) 0.9 (v) 1.85. (vi) 2.4

6. Between which two values of x is the function positive?

7. Write down the two values of x corresponding to each of the following values of the function.

(i) -10.5 (ii) -3. (iii) 2. (iv) 4.25.

GRAPH OF $5 + 2x - 4x^2$



Example (6)

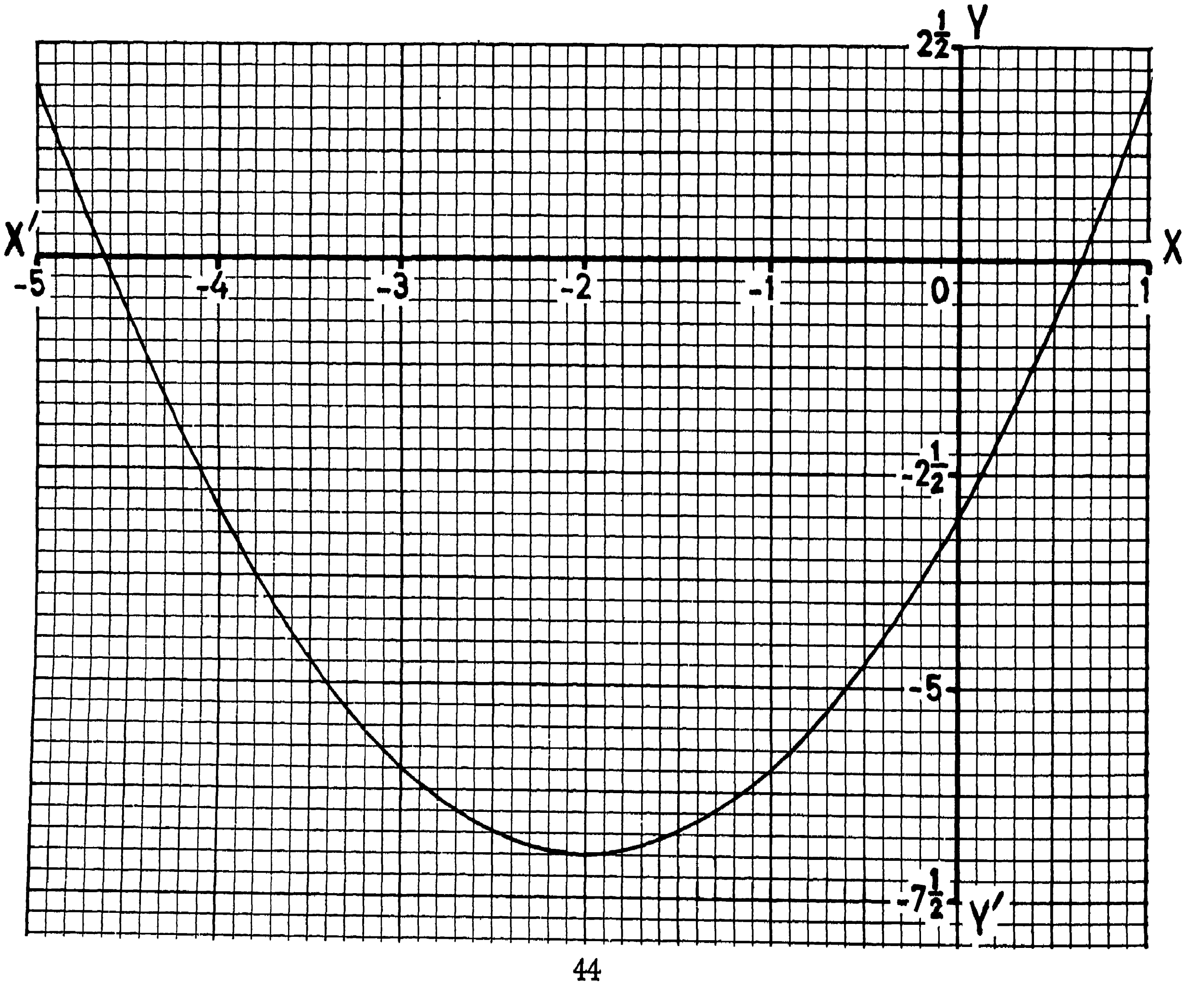
The figure below shows the graph of the function $x^2 + hx + k$ where h and k are constants.

Exercises

Use the graph to work the following

1. State the lowest value of the function.
2. Write down the corresponding value of x
3. Read off to two decimal places the values of x where the graph crosses the x-axis
4. From the intercept on the y-axis write down the value of k
5. Using the expression $x = -\dfrac{b}{2a}$ for finding the axis of symmetry, calculate the value of h
6. Check the value of h by substituting a suitable pair of values of x and y from the graph in the equation $y = x^2 + hx + k$ (knowing the value of k)
7. Write down the *function* whose graph is shown.

GRAPH OF $x^2 + hx + k$



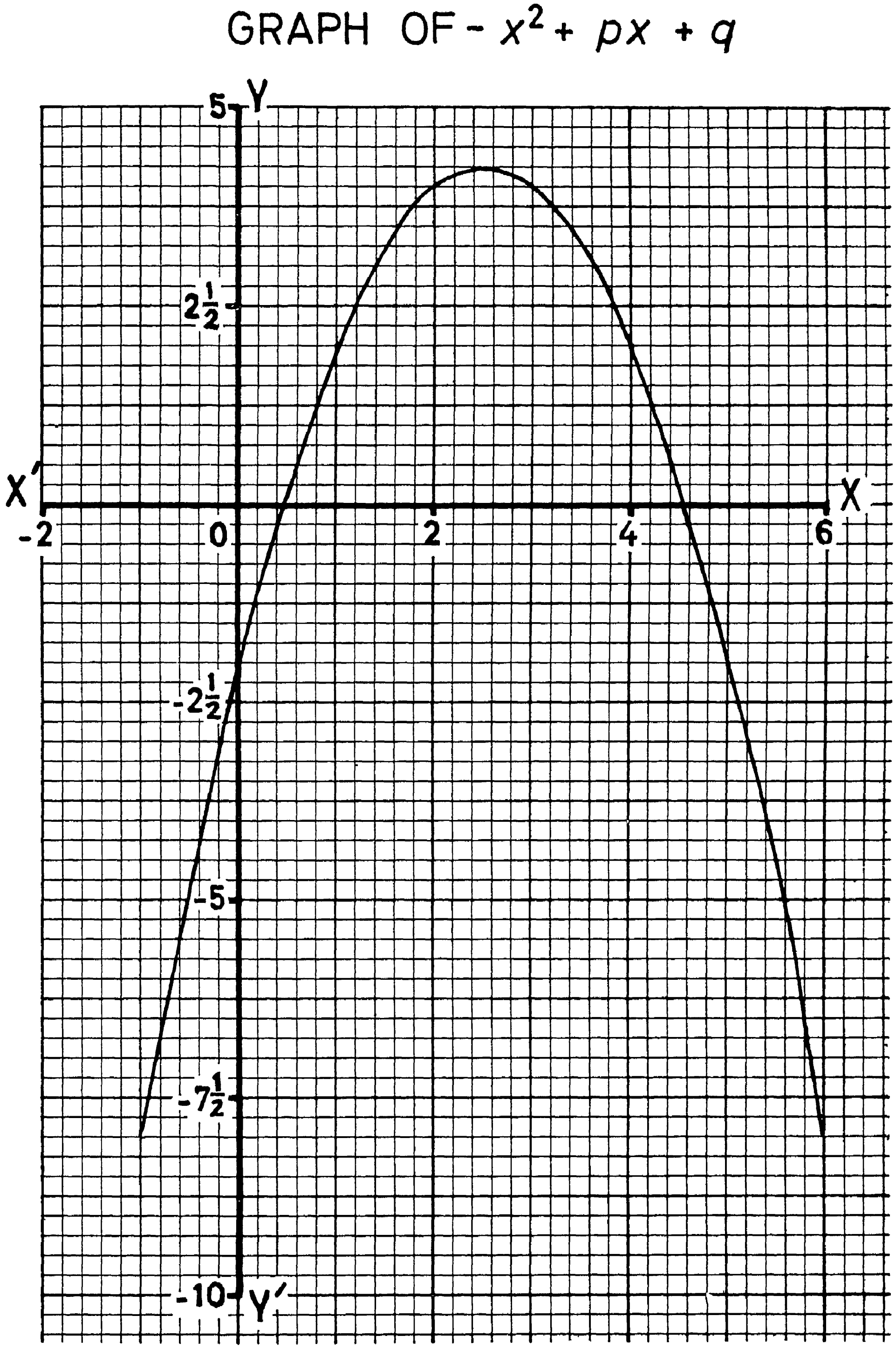
Example (7)

The figure shows the graph of the function $-x^2 + px + q$ where p and q are constants

Exercises

Use the graph to work the following

1. Say how many tenths of an inch represent 1 unit for y.



2. Which value of x gives the maximum value of the function?
3. Write down the maximum value of the function
4. State between which values of x the function is positive.
5. Write down the values of the function for the following values of x

   (i) 4 (ii) 5 (iii) $-0\cdot4$ (iv) $1\cdot8$

6. Write down the two values of x for which the function has each of the following values

   (i) $2\frac{1}{2}$ (ii) -3. (iii) $0\cdot5$

7. Use the methods of the previous example to find the values of p and q in the function

$$-x^2 + px + q$$

8. Write down the function whose graph is shown

Hints on Drawing Graphs of Quadratic Functions

1 *Choice of values*

If $y = \mathrm{f}(x)$ and the range of values for x is not specified, select seven or eight integral values of x evenly spaced about the axis of symmetry (if possible) and calculate the corresponding values of y, as in Chapter 5, Example (1)

2. *Use of axis of symmetry*

If the values of x are so spaced, the values of y will be symmetrical and will give a guide to the minimum (or maximum) value of y

3. *Scale*

Normally choose as large a scale as your graph sheet will allow If the scale is too small the curve will be easy to draw since the plotted points will be close together, but a small error in drawing will represent a relatively large error in unit value But if the scale is very large the resultant gain in unit accuracy may be offset by errors in drawing caused by the points being too far apart.

4 *Drawing*

(*a*) Draw very light lines at first so that they can easily be erased and redrawn if necessary
(*b*) Work from the inside of the curve
(*c*) Complete the part around the turning point last *

5 *Additional values*

For greater accuracy additional values of x around the turning point may be plotted

6 *The constant term*

Make use of the constant term in checking where the graph should cross the y-axis

7 *Minimum and maximum values*

(*a*) The graph will have a minimum value if the coefficient of x^2 is positive.
(*b*) The graph will have a maximum value if the coefficient of x^2 is negative

* The point on the graph of a quadratic function where the function has a maximum or minimum value is called a **turning point**

Exercises on Chapter 5

1. On separate graph sheets draw the graphs of the following functions:

	Function	Range of Values for x	No of Units to 1 inch For x	For y
(*a*)	$x^2 + 4x + 1$	from -5 to $+1$	1	2
(*b*)	$x^2 - 5x + 3$	from -1 to $+6$	2	$2\frac{1}{2}$
(*c*)	$-x^2 + 3x + 7$	from -3 to $+6$	2	5
(*d*)	$-x^2 - 2x + 5$	from -5 to $+3$	1	$2\frac{1}{2}$
(*e*)	$2x^2 - 7x - 4$	from -1 to $+5$	1	$2\frac{1}{2}$
(*f*)	$-3x^2 + 15x + 2$	from -1 to $+5$	1	5

From the graphs in each case, find

(i) The turning value (maximum or minimum) of the function.
(ii) Whether your graph is symmetrical about the line parallel to the y-axis through this turning point
(iii) Whether your graph gives an intercept on the y-axis equal to the constant term of the function
(iv) Between which values of x the function is (a) increasing, (b) decreasing.
(v) The range of values of x for which the function is (a) negative for (*a*), (*b*), and (*e*), (b) positive for (*c*), (*d*), and (*f*)

2. Draw the graph of $x^2 + 3x - 4$ choosing suitable values for x and suitable scales Plot at least seven points on the graph. From your graph find

(*a*) The minimum value of $x^2 + 3x - 4$.
(*b*) The two ranges of values of x for which the value of the expression lies between -2 and -4

3. Draw the graph of $7 + x - 2x^2$ as in Exercise **2.** Make use of your graph to determine

(*a*) The maximum value of $7 + x - 2x^2$
(*b*) Between which values of x the function $7 + x - 2x^2$ is positive.

Chapter 6

THE QUADRATIC EQUATION

In the previous chapter we saw from their graphs how certain quadratic functions behaved as x varied continuously over a limited range of values We will now examine how such graphs may be used to solve quadratic equations by two methods.

Solution of Quadratic Equations—First Method

Example (1)

The figure opposite shows the graph of $y = x^2 - 3x$ from $x = -2$ to $x = 5$

Exercises

Use the graph to complete the following

1. (*a*) $y = x^2 - 3x$ Therefore, to solve the equation $x^2 - 3x = 0$ we find the values of x for which $y =$

(*b*) This occurs where the graph crosses (say which line)

(*c*) The roots of $x^2 - 3x = 0$ are therefore $x =$ and $x =$

2. (*a*) To solve the equation $x^2 - 3x = 4$ we find the values of x for which $y =$

(*b*) The roots of $x^2 - 3x = 4$ are $x =$ and $x =$

3. (*a*) To solve the equation $x^2 - 3x = -2$ we find the values of x for which $y =$

(*b*) The roots of $x^2 - 3x = -2$ are $x =$ and $x =$

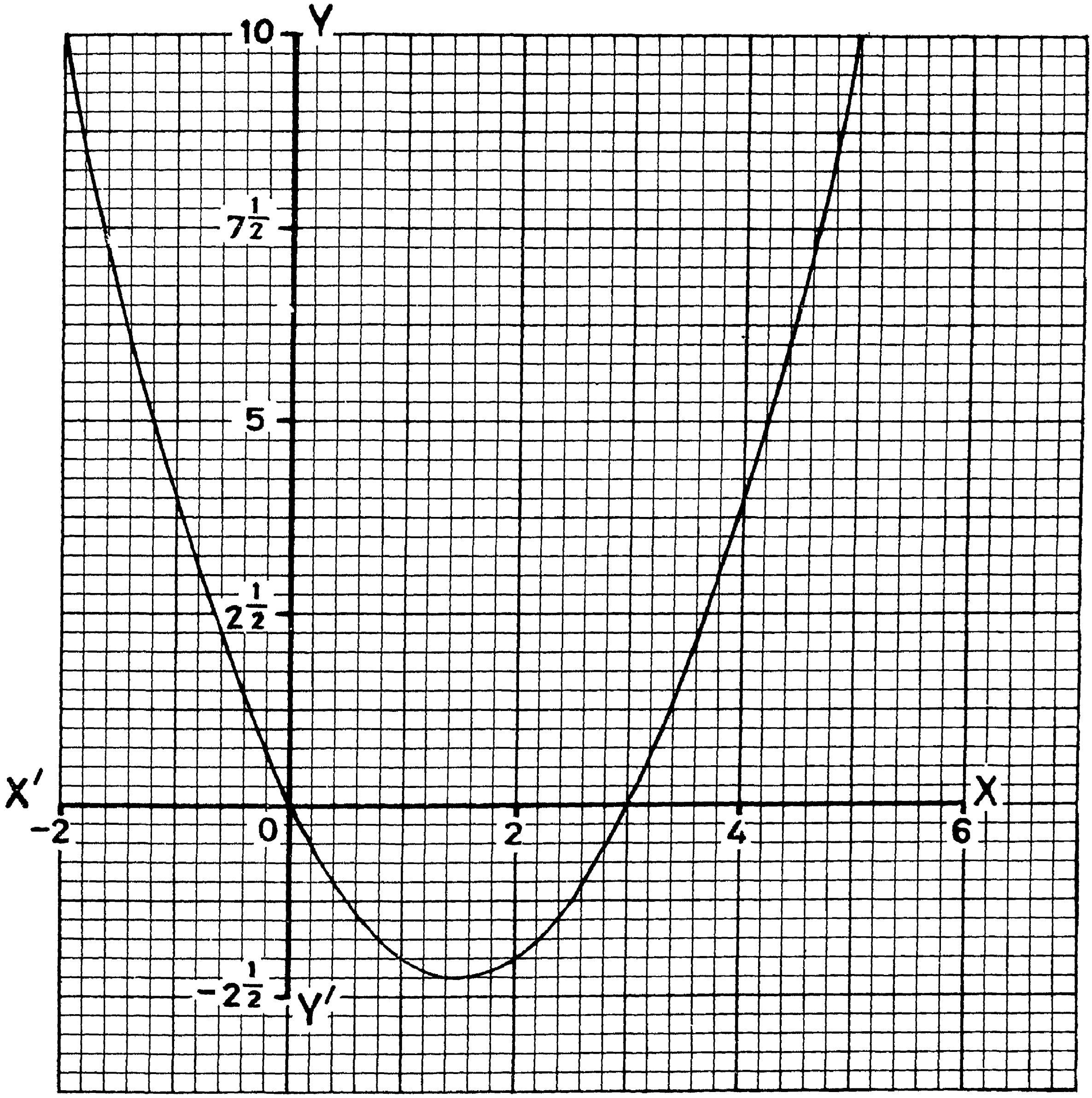
4. The roots of $x^2 - 3x = 7$ are $x =$. and $x =$

5. The roots of $x^2 - 3x = -1\frac{1}{4}$ are $x =$ and $x =$

6. (*a*) The solution of $x^2 - 3x = -2\frac{1}{4}$ is $x =$

(*b*) Explain why there is only one point on the graph for which $x^2 - 3x = -2\frac{1}{4}$

GRAPH OF $y = x^2 - 3x$



Example (2)

The figure opposite shows the graph of $y = 2x^2 - 5x - 3$ from $x = -2$ to $x = 4$

Exercises

Use the graph to complete the following

1. (*a*) $y = 2x^2 - 5x - 3$ Therefore, to solve the equation $2x^2 - 5x - 3 = 0$ we find the values of x for which $y =$ ..

(*b*) The roots of the equation $2x^2 - 5x - 3 = 0$ are therefore $x =$. and $x =$

2. (*a*) To solve the equation $2x^2 - 5x - 3 = 4$ we find the values of x for which $y =$

(*b*) These values are $x =$ and $x =$

(*c*) Transposing the number 4 to the left side of the equation we see that these values are the roots of the equation . $= 0$

3. (*a*) The roots of the equation $2x^2 - 5x - 3 = 9$ are $x =$ . . and $x =$ . .

(*b*) These are also the roots of the equation $= 0$

4. (*a*) The roots of the equation $2x^2 - 5x - 3 = -3$ are $x =$ . . and $x =$ ..

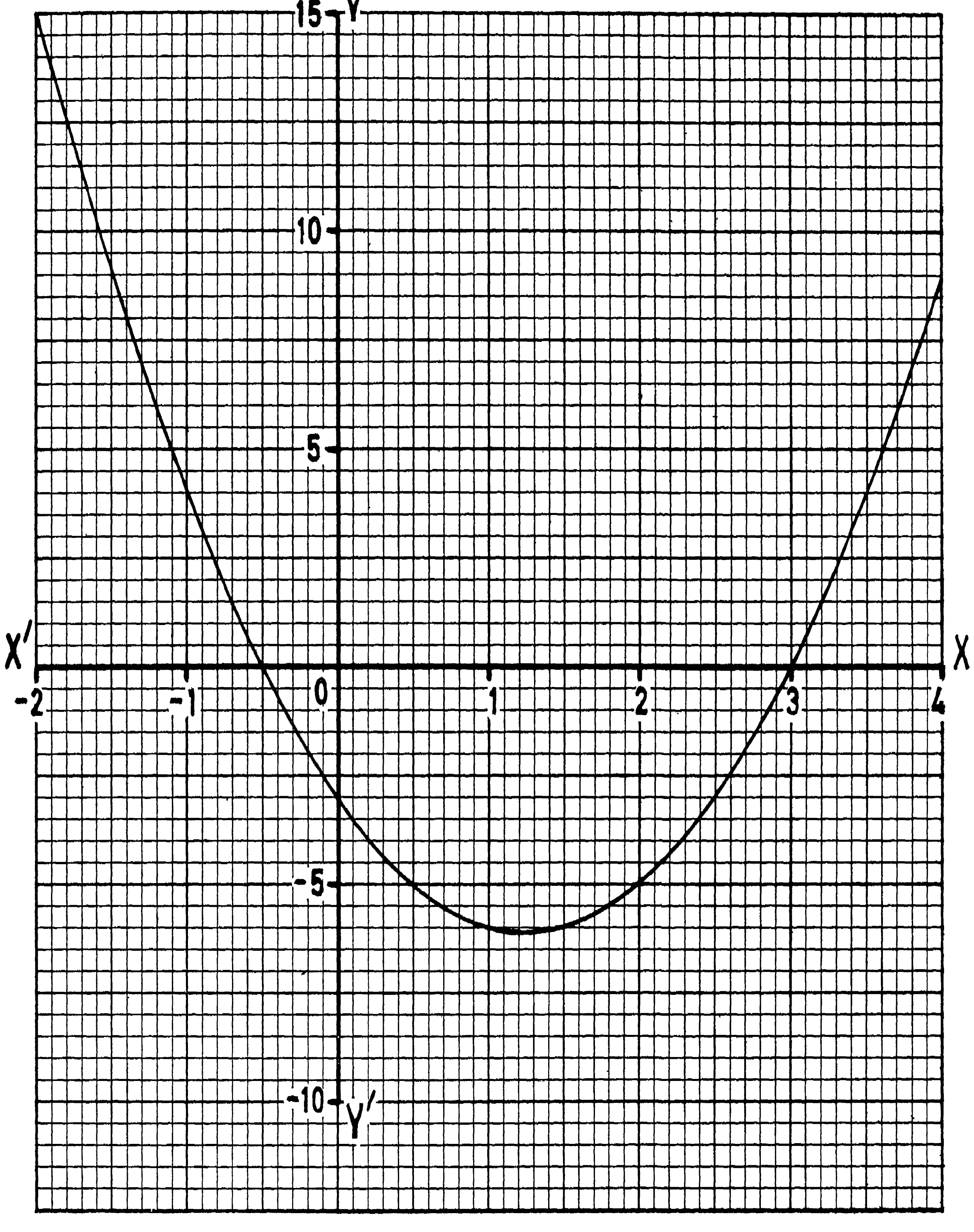
(*b*) These are also the roots of the equation $= 0$

5. (*a*) The equation for which there is only one root* is $2x^2 - 5x - 3 =$ . . and that root is $x =$

(*b*) Say at which point on the graph this occurs

* Mathematicians say there are two equal roots

GRAPH OF $y = 2x^2 - 5x - 3$



Example (3)

The figure opposite shows the graph of $y = x^2 + x - 4$ from $x = -4$ to $x = 3$.

Exercises

Use the graph to complete the following

1. The roots of $x^2 + x - 4 = 0$ are $x =$. and $x =$

2. The roots of $x^2 + x - 4 = 2$ are $x =$ and $x =$.

3. The roots of $x^2 + x - 4 = -2$ are $x =$ and $x =$

4. State how many distinct roots the equation $x^2 + x - 4 = k$ has

(*a*) If k is positive
(*b*) If $k = -3$
(*c*) If $k = -4\frac{1}{4}$

5. The minimum value of $x^2 + x - 4$ is Therefore, there are no roots of

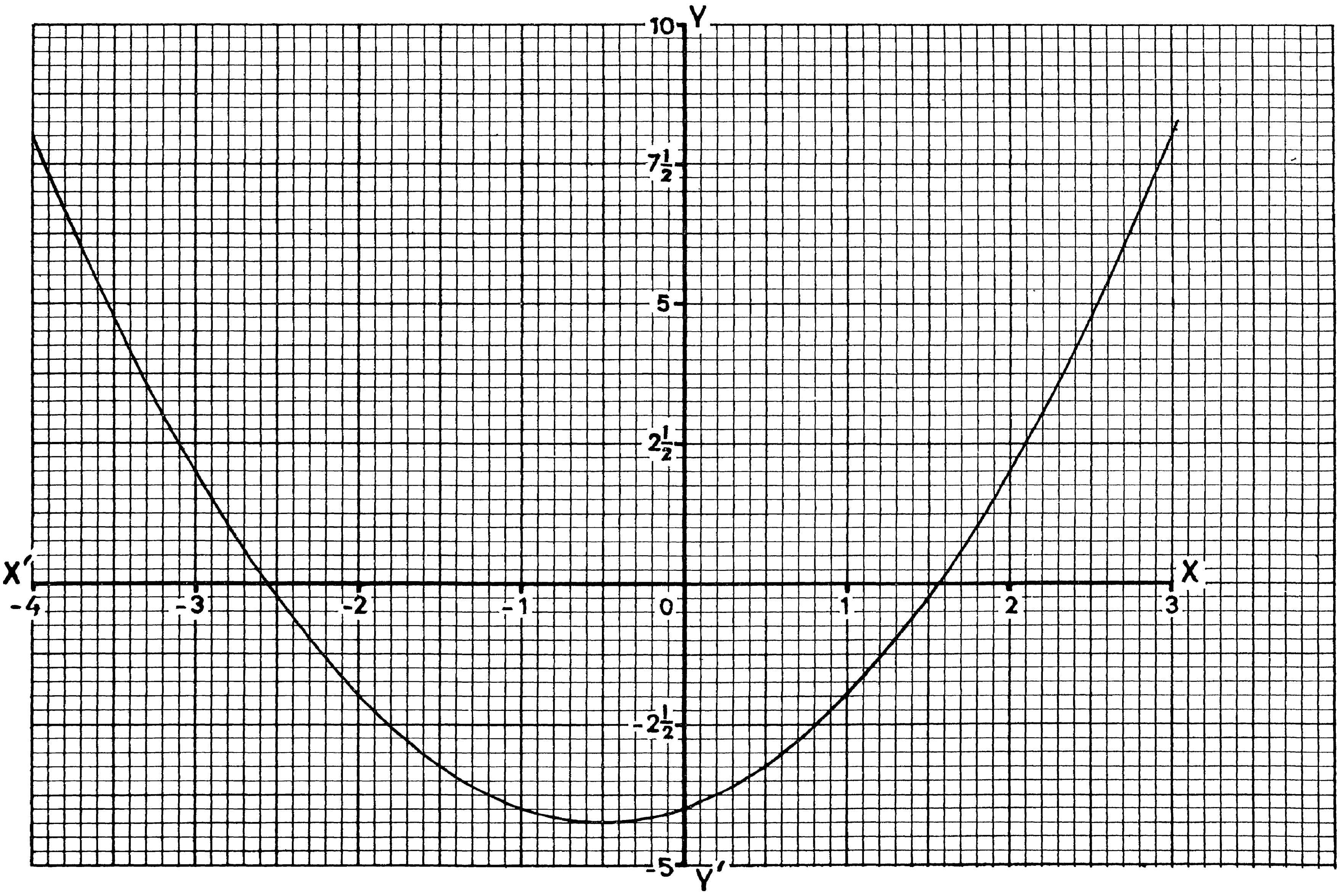
$$x^2 + x - 4 = k$$

if k is less than

6. Summing up, the equation $x^2 + x - 4 = k$ has

(*a*) Two distinct roots if $k >$
(*b*) One distinct root (that is, two equal roots) if $k =$ ..
(*c*) No roots if $k <$

GRAPH OF $y = x^2 + x - 4$



Example (4)

The figure opposite shows the graph of $y = -x^2 + 4x + 2$ from $x = -2$ to $x = 6$.

Exercises

Use the graph to complete the following

1. Along the y axis there are units to 1 inch Therefore, 1 unit = 0 . . inch

2. The roots of $-x^2 + 4x + 2 = 0$ are $x =$. and $x =$.

3. The roots of $x^2 - 4x - 2 = 0$ [that is, of $-(-x^2 + 4x + 2) = 0$] are $x =$. and $x =$ . .

4. The roots of $-x^2 + 4x + 2 = -3$ are $x =$. and $x =$ . ..

5. The roots of $-x^2 + 4x + 2 = 5$ are $x =$. and $x =$ ..

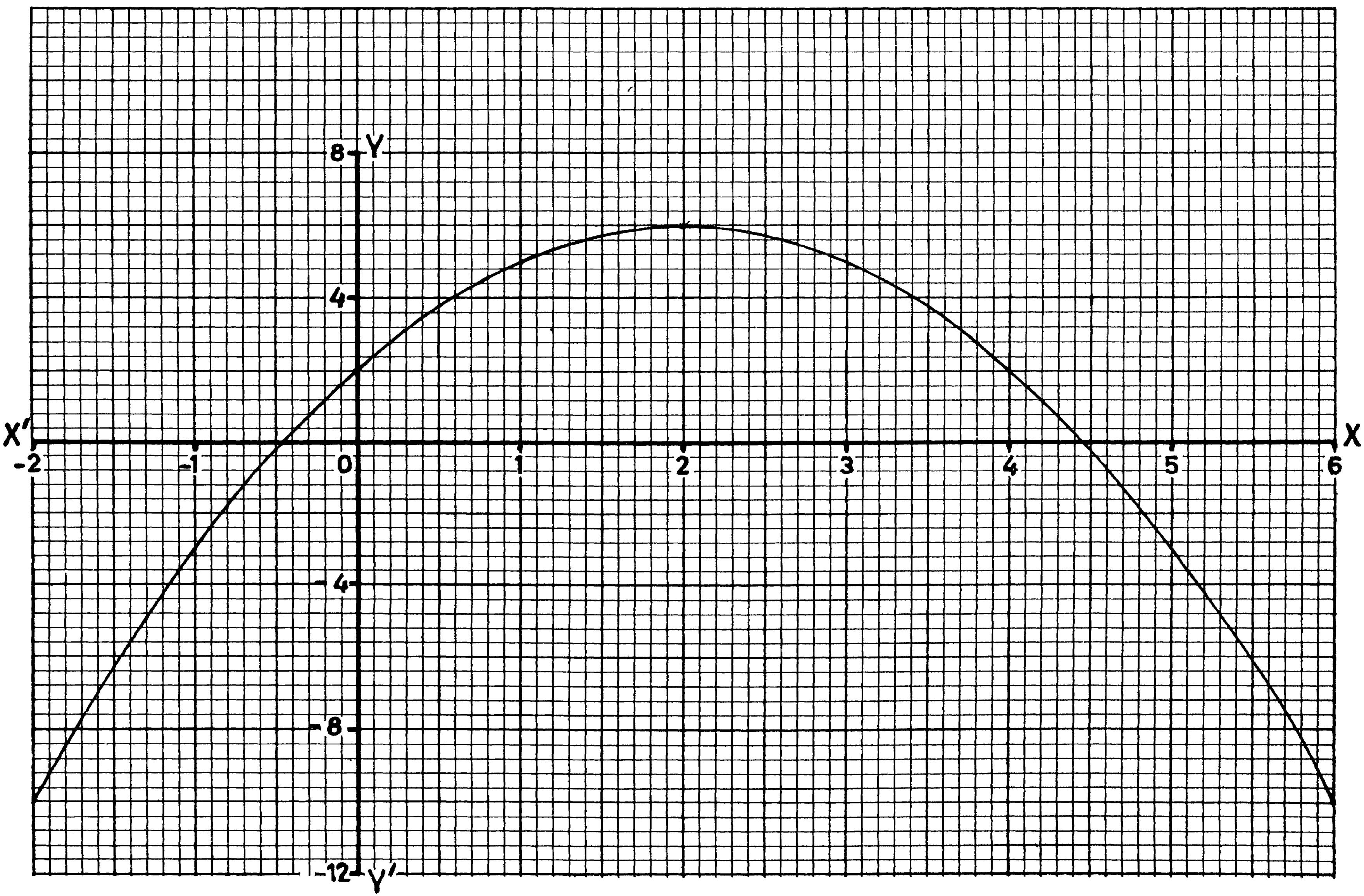
6. The value of x for which $-x^2 + 4x + 2 = 6$ is $x =$.

7. Are there any roots of the equation $-x^2 + 4x + 2 = 7$?

8. The equation $-x^2 + 4x + 2 = k$ has

(*a*) Two distinct roots if $k <$.
(*b*) One distinct root (that is, two equal roots) if $k =$ ..
(*c*) No roots if $k >$ . .

GRAPH OF $y = -x^2 + 4x + 2$



Example (5)

The figure opposite shows the graph of $y = 4x^2 + 4x - 1$ from $x = -4$ to $x = 3$

Exercises

Use the graph to complete the following

1. The roots of $4x^2 + 4x - 1 = 0$ are $x =$ and $x =$

2. The roots of $4x^2 + 4x - 1 = 34$ are $x =$ and $x =$

3. The roots of $4x^2 + 4x - 1 = 10$ are $x =$ and $x =$.

4. The roots of $4x^2 + 4x - 1 = -1$ are $x =$ and $x =$

5. The value of x for which $4x^2 + 4x - 1 = -2$ is $x =$

6. If $4x^2 + 4x = 24$, then $4x^2 + 4x - 1 =$, and therefore $4x^2 + 4x = 24$ if $x =$ or $x =$

7. If $4x^2 + 4x = 8$, then $4x^2 + 4x - 1 =$, and therefore $4x^2 + 4x = 8$ if $x =$ or $x =$

8. If $4x^2 = 6 - 4x$, then $4x^2 + 4x =$. , and $4x^2 + 4x - 1 =$, and therefore $4x^2 = 6 - 4x$ if $x =$. or $x =$

9. If $4x^2 = 3 - 4x$, then $4x^2 + 4x =$, and $4x^2 + 4x - 1 =$, and therefore $4x^2 = 3 - 4x$ if $x =$. or $x =$.

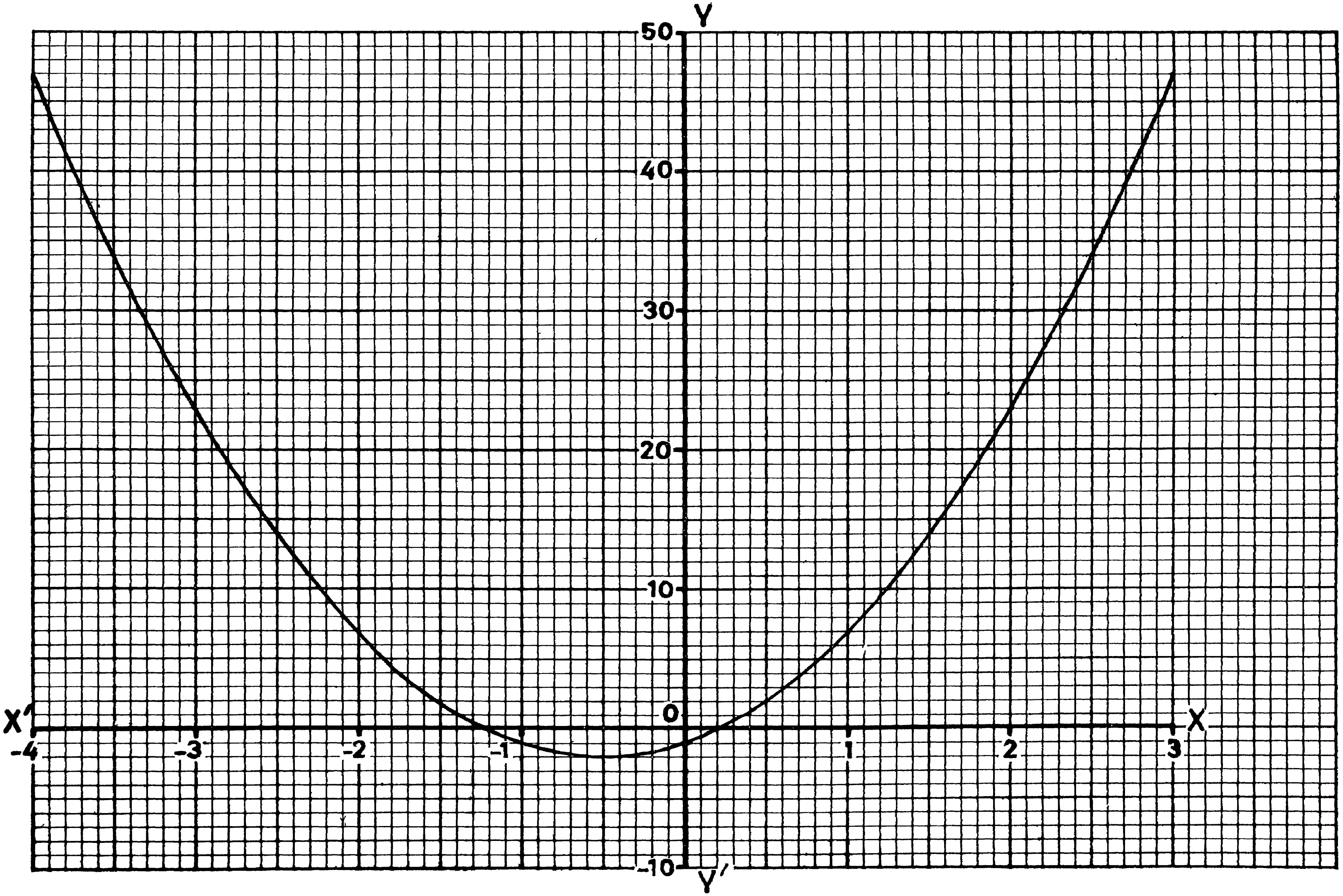
10. If $8x^2 + 8x = 57$, then $4x^2 + 4x =$ . . , and $4x^2 + 4x - 1 =$, and therefore $8x^2 + 8x = 57$ if $x =$ or $x =$.

11. If $2x^2 + 2x = 7\frac{1}{2}$, then $4x^2 + 4x - 1 =$, and therefore $2x^2 + 2x = 7\frac{1}{2}$ if $x =$ . . or $x =$

12. By eliminating x^2 between $y = 4x^2 + 4x - 1$ and $0 = 4x^2 + 5x - 2$, determine the equation of the straight line whose intersections with the given graph would enable you to solve $4x^2 + 5x - 2 = 0$

13. Find similarly the equation of the straight line whose intersections with the given graph would enable you to solve $4x^2 + 6x + 1 = 0$

GRAPH OF $y = 4x^2 + 4x - 1$



Example (6)

The figure opposite shows the graph of $y = -2x^2 + 3x + 5$ from $x = -2$ to $x = 4$

Exercises

Use the graph to work the following

1. Solve, correct to two decimal places (where necessary)

(*a*) $-2x^2 + 3x + 5 = 0$
(*b*) $-2x^2 + 3x + 5 = -5$
(*c*) $-2x^2 + 3x + 5 = 2.5$
(*d*) $-2x^2 + 3x + 5 = 6\frac{1}{8}$
(*e*) $-2x^2 + 3x + 14 = 0$
(*f*) $-2x^2 + 3x + 4.5 = 0$
(*g*) $-2x^2 = -3x - 1$
(*h*) $-4x^2 + 6x + 19 = 0.$
(*i*) $-4x^2 + 6x = 0.$
(*j*) $-x^2 + 1.5x + 3 = 0$

2. Say for which values of k the equation $-2x^2 + 3x + 5 = k$ has

(*a*) Two distinct roots
(*b*) Two equal roots
(*c*) No roots

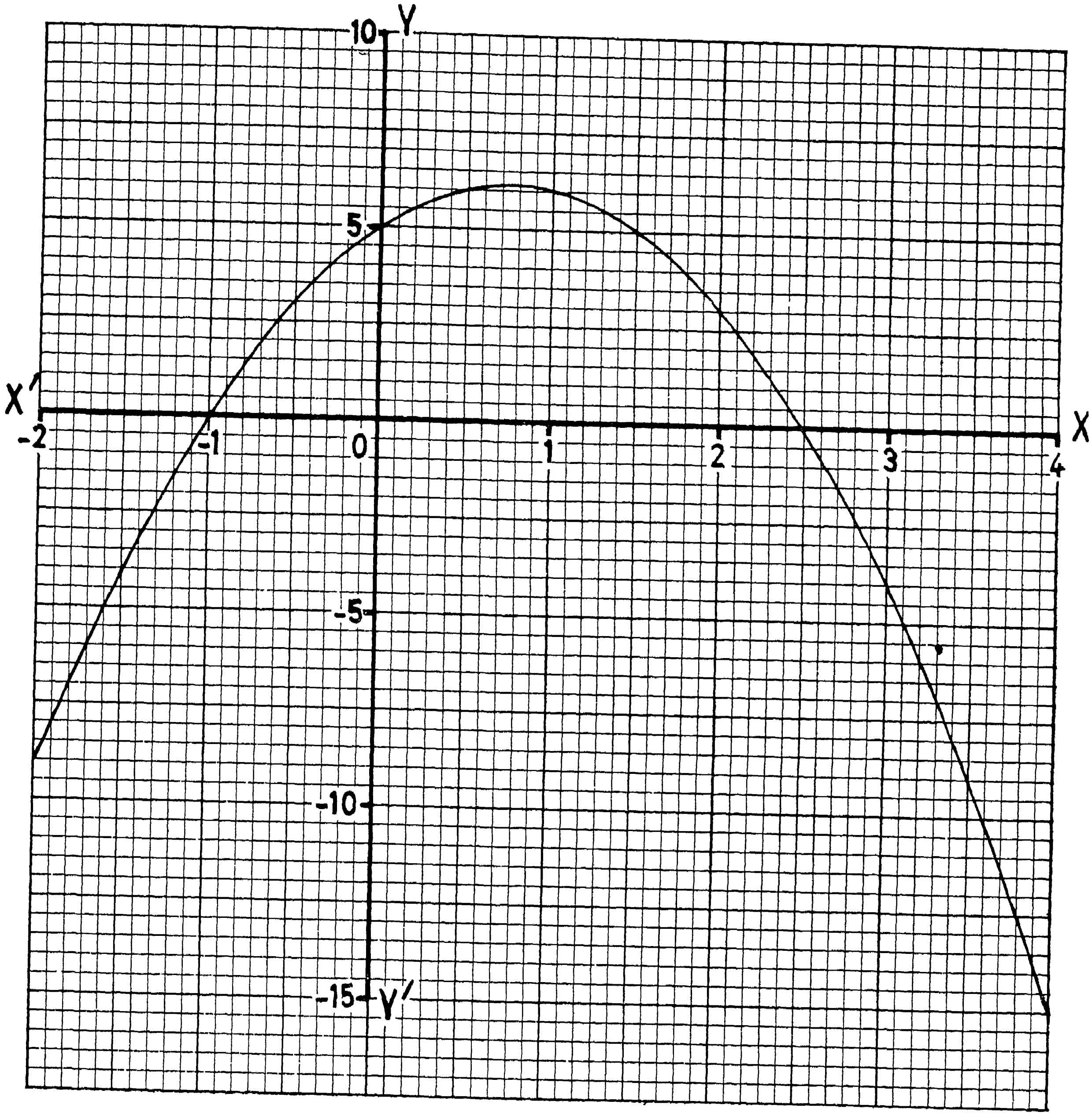
3. If $x = 2$ is one root of an equation $-2x^2 + 3x + 5 = k$ find

(*a*) The value of k.
(*b*) The other root of this equation.

4. On a separate graph sheet copy the graph of $y = 2x^2 + 3x + 5$.
By drawing suitable straight line graphs on your graph sheet, solve the equations

(*a*) $-2x^2 - x + 2 = 0$
(*b*) $-2x^2 + 4x + 3 = 0$
(*c*) $2x^2 - 5x - 6 = 0$

GRAPH OF $y = -2x^2 + 3x + 5$



Example (7)

The figure opposite shows the graph of $y = 5x^2 + 2x - 4$ from $x = -2$ to $x = 2$

Exercises

Use the graph to work the following

1. Solve, correct to two decimal places (where necessary)

(*a*) $5x^2 + 2x - 4 = 0$
(*b*) $5x^2 + 2x - 4 = 12$
(*c*) $5x^2 + 2x - 4 = -2\ 8$
(*d*) $5x^2 + 2x - 12 = 0$
(*e*) $5x^2 + 2x - 3 = 0$
(*f*) $5x^2 + 2x + 0\ 2 = 0$
(*g*) $5x^2 = 5 - 2x$
(*h*) $5x^2 + 2x = 15$

2. State for which values of t the equation $5x^2 + 2x - 4 = t$ has

(*a*) Two distinct roots
(*b*) Two equal roots
(*c*) No roots.

3. Solve

(*a*) $10x^2 + 4x - 8 = 12$
(*b*) $10x^2 + 4x = 5$
(*c*) $2\ 5x^2 + x - 2 = 0\ 75$

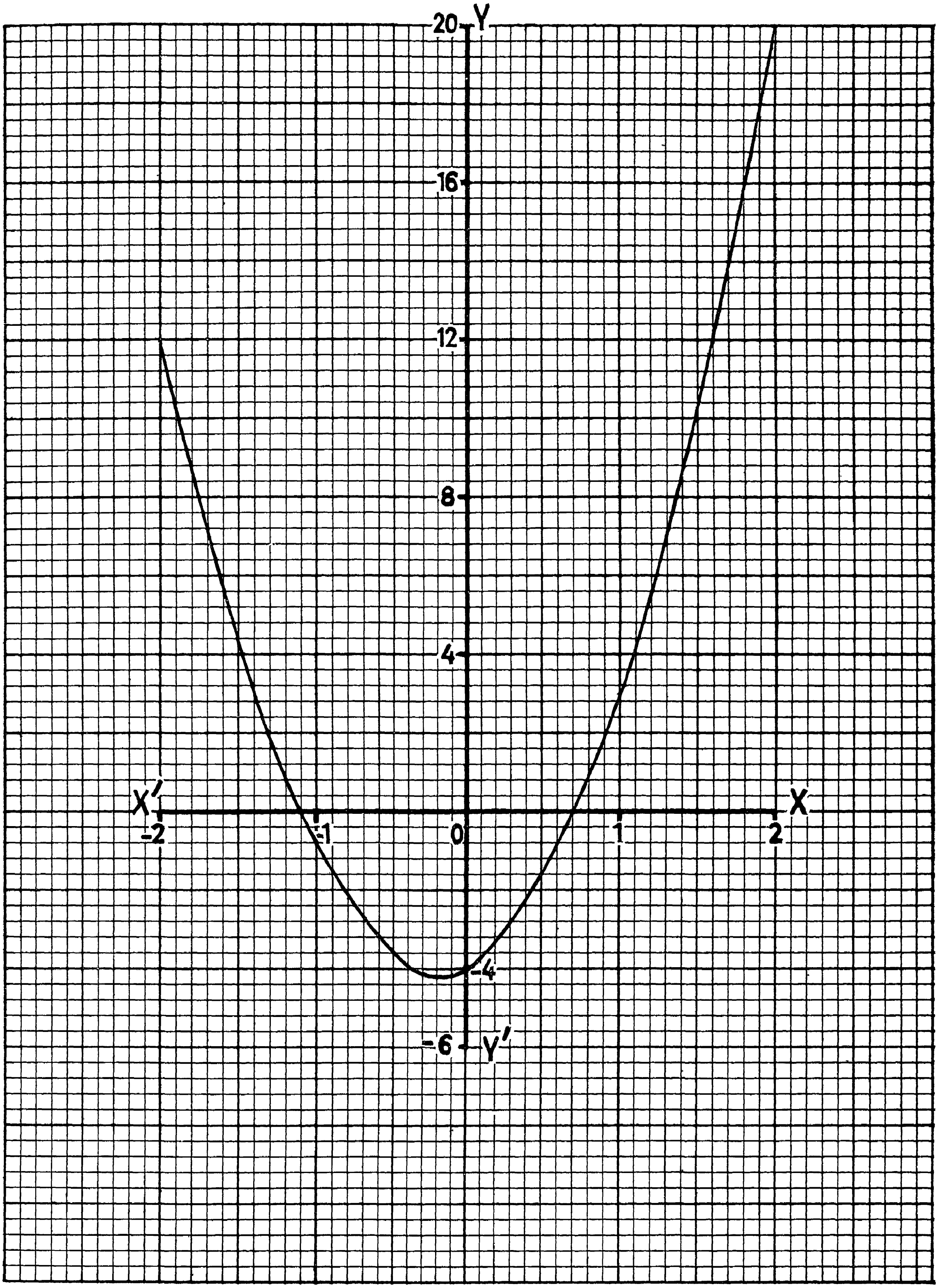
4. If one of the roots of $5x^2 + 2x - 4 = t$ is $x = 1\ 05$, find t and so write down the other root

5. Use algebraic methods to check your answers to **1** (*e*) and **1** (*a*)

6. On a separate graph sheet copy the graph of $y = 5x^2 + 2x - 4$
From the intersections of suitable straight lines with your graph solve

(*a*) $5x^2 + x - 5 = 0$
(*b*) $5x^2 + 6x - 2 = 0$
(*c*) $5x^2 + x - 1 = 0$

GRAPH OF $y = 5x^2 + 2x - 4$



Solution of Quadratic Equations—Second Method

Example (8)

Any quadratic equation which has roots can be solved from the graph of $y = x^2$ and the appropriate straight line graph Thus to solve the equation $2x^2 - x - 6 = 0$ we first write $2x^2 = x + 6$ and then $x^2 = \frac{x+6}{2}$ We now draw (opposite) the graphs of $y = x^2$ and $y = \frac{x+6}{2}$ (The continuous straight line graph)

Exercises

Use the graph to answer the following

1. At which points on these graphs will the values of x^2 and $\frac{x+6}{2}$ be equal?

2. For which values of x will $x^2 = \frac{x+6}{2}$?

3. Hence, complete the statement "The roots of the equation $2x^2 - x - 6 = 0$ are $x =$. and $x =$."

4. Taking the values $x = -2$, $x = 0$, $x = 1$, check that the dotted line is the graph of $y = \frac{-3x+2}{5}$.

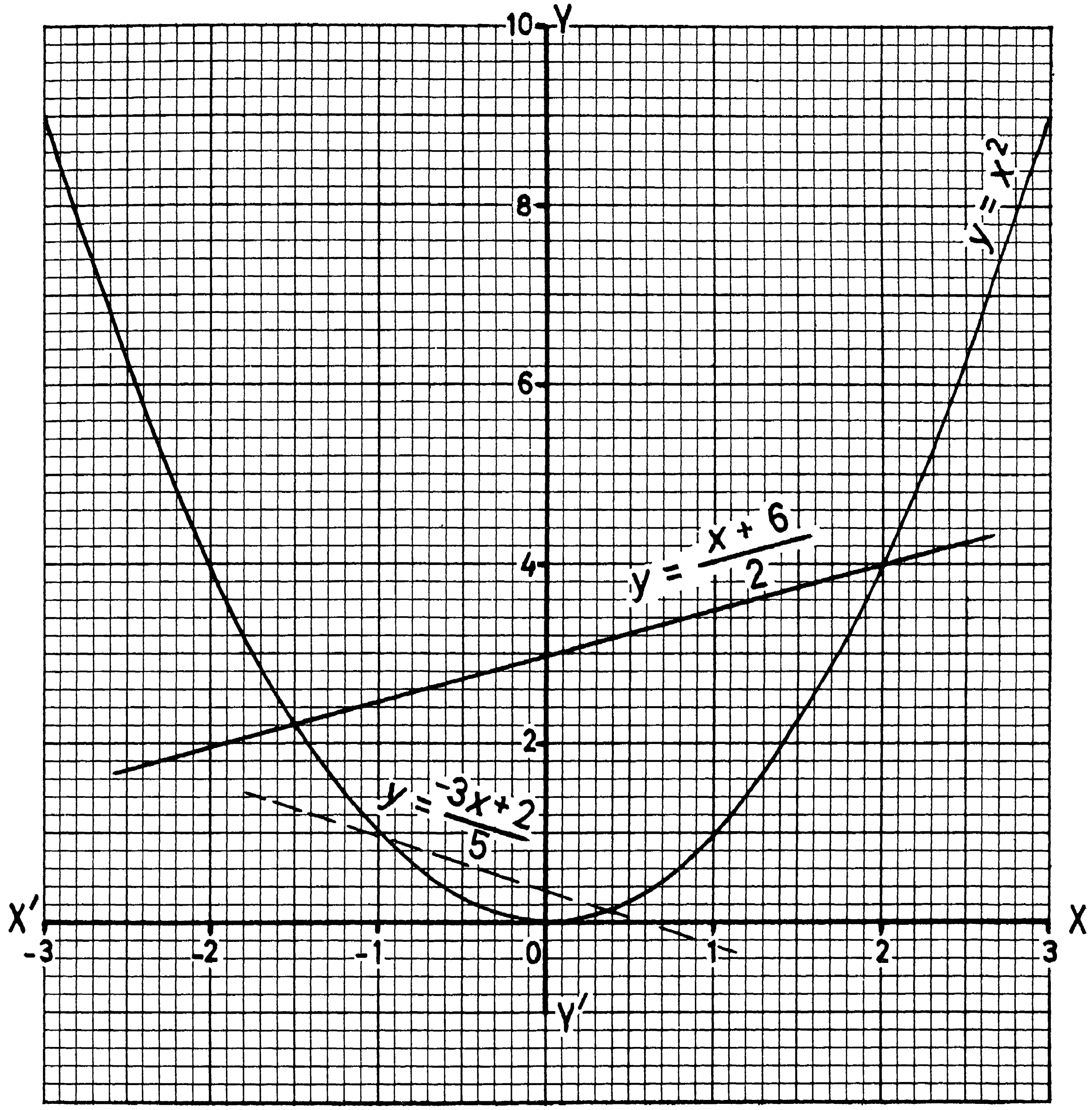
5. (*a*) For which values of x will $x^2 = \frac{-3x+2}{5}$?

(*b*) Hence, the roots of the equation (in simplest form) are $x =$ and $x =$

6. Say which straight line graphs you would draw (with the graph of $y = x^2$) to solve the following equations

(*a*) $x^2 - 5x + 6 = 0$
(*b*) $x^2 + 7x - 8 = 0$
(*c*) $2x^2 = 5x + 2$
(*d*) $2x^2 - 7x = 3$
(*e*) $3x^2 + 6x - 1 = 0$
(*f*) $5x^2 - 9x + 2 = 0$

GRAPHS OF $y = x^2$, $y = \frac{x+6}{2}$, and $y = \frac{-3x+2}{5}$



Example (9)

On the opposite page are shown the graph of $y = x^2$ and of four parallel straight line graphs The equation of one of these lines is given as $y = 2x + 3$

Exercises

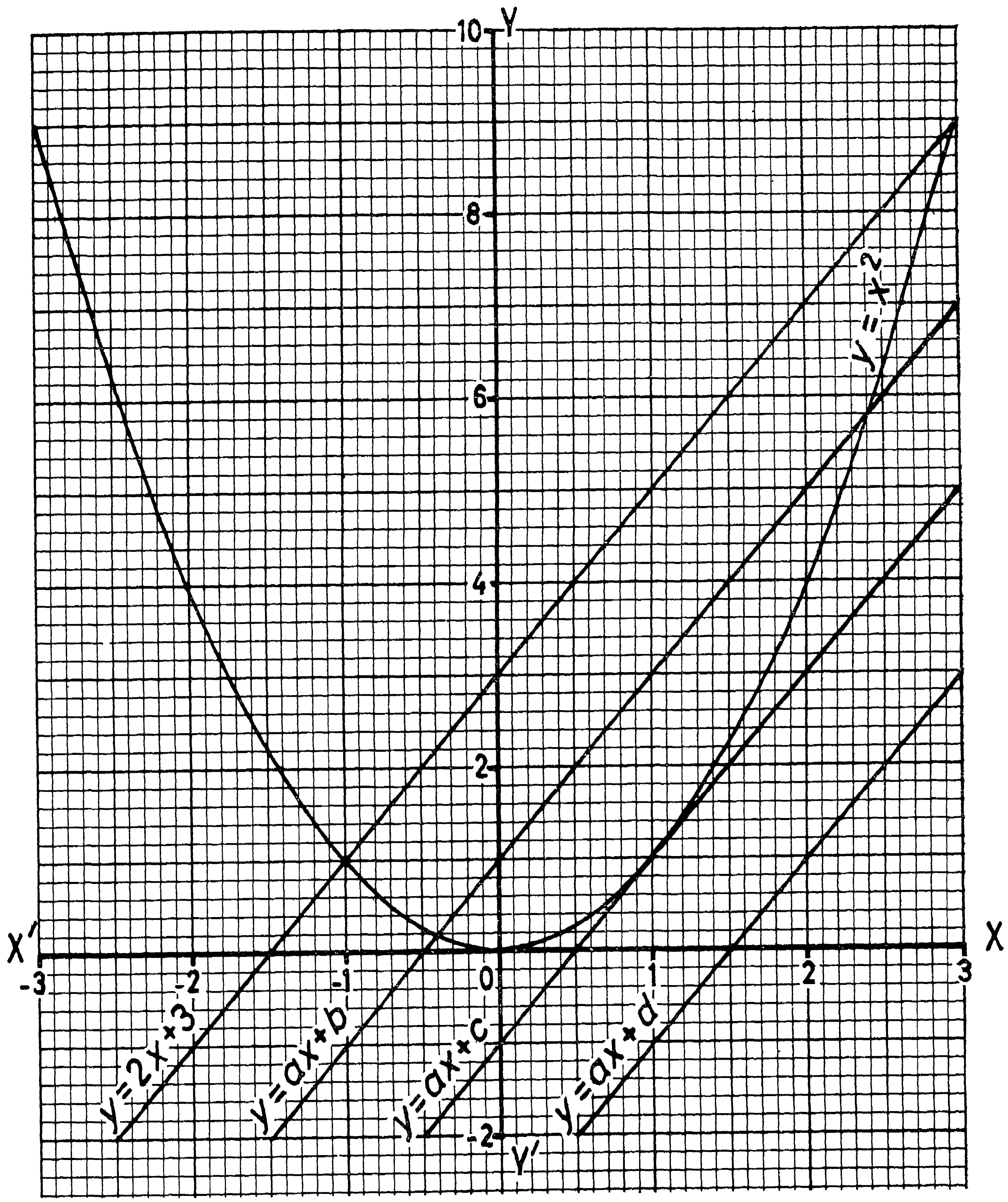
Answer the following from the graphs

1. From your knowledge of parallel line graphs and intercepts on the y-axis, find the values of a, b, c, and d in the equations (i) $y = ax + b$, (ii) $y = ax + c$, (iii) $y = ax + d$, and rewrite these equations
2. The roots of the equation $x^2 - 2x - 3 = 0$ are $x =$ and $x =$
3. Give to two decimal places the roots of the equation $x^2 = ax + b$, that is, of the equation $x^2 = \ . \ x +$
4. Explain why the equation $x^2 = ax + c$, that is, $x^2 = \quad x + \quad$ has two equal roots and state its solution
5. Explain why the equation $x^2 = ax + d$, that is, $x^2 = \quad x + \quad$ has no roots
6. To summarize, the equation $x^2 - 2x - k = 0$

   (*a*) Has two distinct roots if $k >$
   (*b*) Has two equal roots if $k =$
   (*c*) Has no roots if $k <$

7. (*a*) Say which straight line graph you would draw to solve $x^2 + 2x - 3 = 0$
   (*b*) State, without drawing the graph, the x-co-ordinates of the points of intersection of your graph with $y = x^2$

GRAPHS OF $y = x^2$, $y = 2x + 3$, etc



Exercises on Chapter 6

1. Draw the graph of $x^2 - 2x$ from $x = -4$ to $x = 6$ and use it to solve the following equations

(i) $x^2 - 2x = 0$ (ii) $x^2 - 2x = 24$ (iii) $x^2 - 2x - 15 = 0$
(iv) $x^2 - 2x = 4$ (v) $x^2 - 2x + 1 = 0$ (vi) $2x^2 - 4x + 1 = 0$

2. Draw the graph of $y = x^2 - 7x + 12$ from $x = 0$ to $x = 7$ and use it to solve the following equations

(i) $x^2 - 7x + 12 = 0.$ (ii) $x^2 - 7x + 12 = 6$ (iii) $x^2 - 7x = 0$
(iv) $x^2 - 7x + 7 = 0$ (v) $x^2 - 7x + 10 = 0$ (vi) $x^2 - 7x + 12 = -\frac{1}{4}$

Rewrite equation (vi) in the form $ax^2 + bx + c = 0$ where a, b, c are whole numbers

3. By drawing the graph of $y = 2x^2 + 3x$ from $x = -4$ to $x = 2$, solve the following equations:

(i) $2x^2 + 3x = 20$ (ii) $2x^2 + 3x - 14 = 0$ (iii) $2x^2 + 3x - 5 = 3.$
(iv) $2x^2 = 5 - 3x.$ (v) $16x^2 + 24x + 9 = 0$

Explain from the graph why

(*a*) There is only one solution of $16x^2 + 24x + 9 = 0$
(*b*) There is no solution of $2x^2 + 3x = -2.$

4. Draw the graph of $7 - 6x - 3x^2$ from $x = -4$ to $x = 2$ and use it to solve, where possible, the following equations

(i) $7 - 6x - 3x^2 = -2$ (ii) $-6x - 3x^2 = 7.$
(iii) $7 - 6x - 3x^2 = 9.$ (iv) $7 - 6x = 10 + 3x^2$
(v) $-6x - 3x^2 = 0.$

Now write down the solutions of

(vi) $3x^2 + 6x - 7 = 2,$ (vii) $3x^2 + 6x = 0,$

and describe the graph of $3x^2 + 6x - 7$

5. (*a*) Draw the graph of $4x^2 - 5x$ from $x = -3$ to $x = 3$ Use this graph to solve, where possible, the equations

(i) $4x^2 - 5x = 6$ (ii) $4x^2 - 5x + 1 = 27.$ (iii) $2x^2 - \frac{5}{2}x = 9$
(iv) $8x^2 - 10x = 7.$ (v) $4x^2 = 5x - 1$ (vi) $12x^2 - 15x + 8 = 0$

(*b*) Write down the equation of the form $4x^2 - 5x = t$ whose two roots are equal, and change this to the form $ax^2 + bx + c = 0$ where a, b, and c are whole numbers
(*c*) State the solution of this equation

6. Draw the graph of $x - 5x^2$ from $x = -3$ to $x = 4$
(*a*) Use this graph to solve, where possible, the equations

(i) $x - 5x^2 = -48$ (ii) $20 + x - 5x^2 = 0$ (iii) $2x - 10x^2 = 7$
(iv) $2x - 10x^2 = 4.$ (v) $x - 5x^2 = 0.05.$ (vi) $5x^2 - x = 10.$

(*b*) Explain for which value(s) of c the equation $x - 5x^2 = c$

(i) Has two distinct roots
(ii) Has two equal roots
(iii) Has no roots

7. Draw the graph of $y = x^2$ from $x = -3$ to $x = 3$ By drawing on the same graph sheet the graphs of:

(i) $y = x + 2$, (ii) $y = 4x - 3$, (iii) $y = 3 - \frac{x}{2}$,

solve the equations

(i) $x^2 = x + 2$ (ii) $x^2 - 4x + 3 = 0$ (iii) $2x^2 + x - 6 = 0$

8. Draw the graph of $y = x^2$ from $x = -4$ to $x = 4$
(*a*) By drawing on the same diagram suitable straight line graphs, solve

(i) $x^2 = \frac{1}{2}(5x + 7)$ (ii) $2x^2 = x + 10$
(iii) $2x^2 + 3x - 15 = 0$ (iv) $5 + 2x - x^2 = 0$

(*b*) Draw also the graph of $y = \frac{1}{2}(2x - 7)$ and use it to explain why there are no solutions of the equation $2x^2 - 2x + 7 = 0$

9. On the same axes, draw from $x = -3$ to $x = 3$ the graphs of $y = x^2$, $y = x + 6$, $y = x - 6$, $y = x - \frac{1}{4}$, $y = -x + 6$, $y = -x - 6$ Use the intersections of the straight line graphs with the graph of $y = x^2$ (where these occur) to discuss the roots of the equations

(i) $x^2 - x - 6 = 0$ (ii) $x^2 - x + 6 = 0$ (iii) $x^2 - x + \frac{1}{4} = 0$.
(iv) $x^2 + x - 6 = 0$ (v) $x^2 + x + 6 = 0$

10. Draw the graph of $y = x + \frac{6}{x} + 5$ from $x = 1$ to $x = 6$

(i) Solve $x + \frac{6}{x} + 5 = 10\frac{1}{2}$

(ii) Solve $x + \frac{6}{x} = 6.8$

(iii) Find the least value of $x + \frac{6}{x}$ when x is positive Hence write down the equation of the form $x + \frac{6}{x} = d$, which has two approximately equal solutions, and solve this equation for x.

Chapter 7

A. THE CUBIC FUNCTION AND THE CUBIC EQUATION

B THE EQUATION $y = \frac{k}{x}$ (THE HYPERBOLA)

1. The Cubic Function and the Cubic Equation

Example (1)

The following table was used in drawing the graphs of the functions x^3 and $x^3 - 4x$ (shown opposite):

x	-3	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3
x^3	-27	-8	$-3\frac{3}{8}$	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	$3\frac{3}{8}$	8	27
$-4x$	12	8	6	4	2	0	-2	-4	-6	-8	-12
$x^3 - 4x$	-15	0	$2\frac{5}{8}$	3	$1\frac{7}{8}$	0	$-1\frac{7}{8}$	-3	$-2\frac{5}{8}$	0	15

Exercises

A Consider the graph of x^3

1. When x is positive/negative x^3 is also positive/negative Therefore, the graph lies in the . and . quadrants
2. State in which two quadrants the graph of $-x^3$ will lie
3. Describe the slope of the graph as you move along it from left to right, noting especially its direction when $x = 0$
4. Use the graph to find approximately the cubes of the following numbers

$$-2{\cdot}6,\ -2{\cdot}1,\ -1{\cdot}6,\ -0{\cdot}7,\ 0{\cdot}9,\ 1{\cdot}4,\ 1{\cdot}9,\ 2{\cdot}5$$

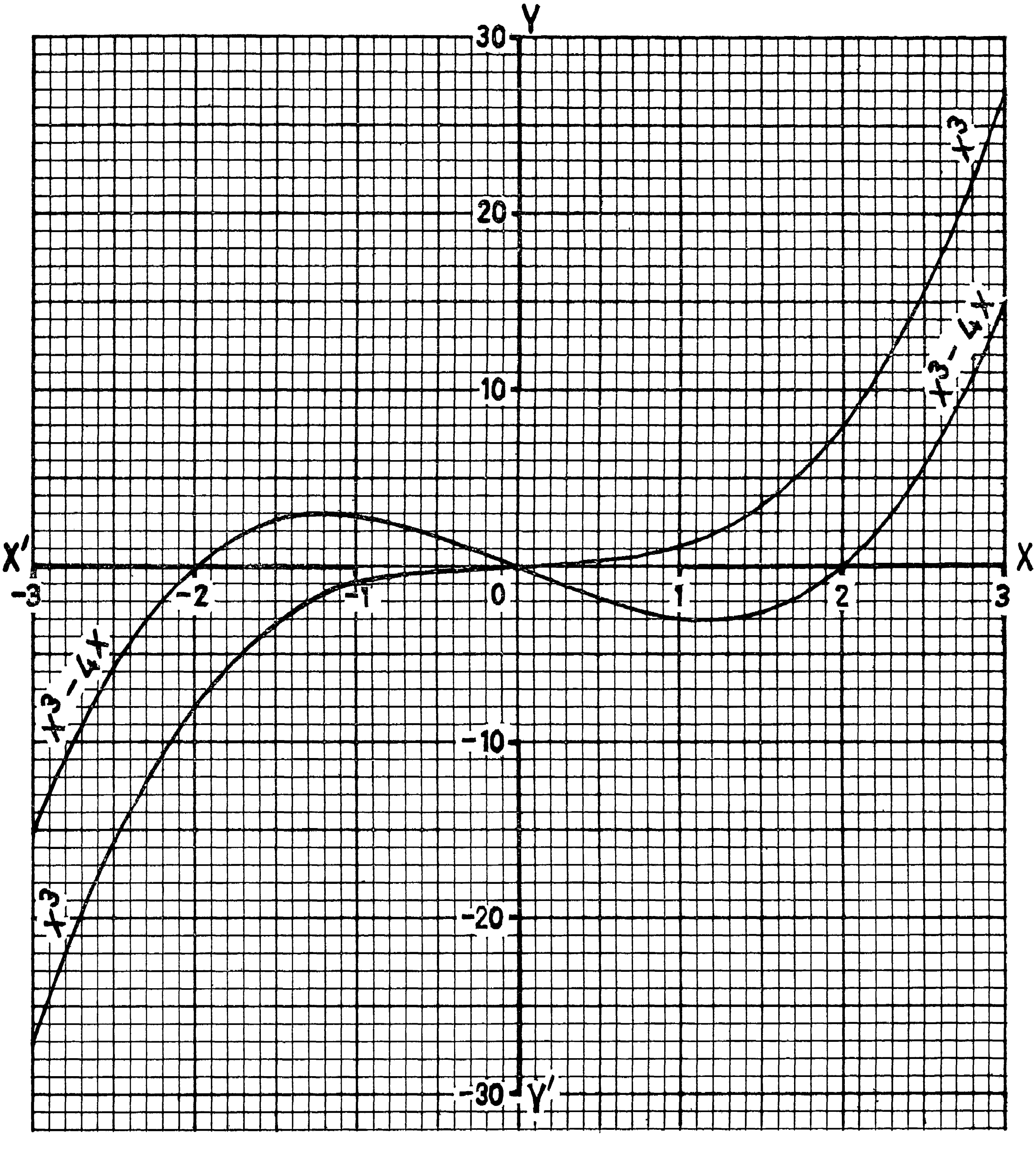
5. Use the graph to find, correct to two decimal places, the cube roots of

$$-21,\ -12,\ -3{\cdot}5,\ -2,\ 1{\cdot}5,\ 8,\ 9{\cdot}5,\ 19,\ 22,\ 25$$

B Consider the graph of $x^3 - 4x$

1. Compare the signs of the terms x^3 and $-4x$ for all values of x
2. (*a*) $x^3 > 4x$ numerically if $x >$ numerically
   (*b*) $4x > x^3$ numerically if x lies between . and
3. Say how the term $-4x$ makes this graph differ from the graph of x^3.

GRAPHS OF x^3 and $x^3 - 4x$



$4x$

4. The graph shows that the function $x^3 - 4x$ is negative if

(a) $x <$.

(b) x lies between . and . ..

5. The graph shows that the function $x^3 - 4x$ is positive if

(a) $x >$ . .

(b) x lies between . . and .

6. Write down the x-co-ordinates of the turning points of the graph
7. Write down the values of $x^3 - 4x$ at these turning points
8. Use the graph to solve $x^3 - 4x = 0$
9. Say how the graph of $x^3 - 4x + 5$ would compare with this graph
10. State, with reasons, whether the graph of $x^3 + 4x$ would have two turning points like the graph of $x^3 - 4x$ or a **point of inflection*** like the graph of x^3

Example (2)

The diagram opposite shows the graph of $2x^3 - 5x + 6$ from $x = -3$ to $x = 3$. In calculating values of the function for plotting, intermediate values of x were used as well as integral values, for example, $x = -1\ 5, -0\ 5, 0\ 5, 1\ 5$

Exercises

Use the graph to answer the following

1. Above which value of x is the function positive?
2. As x increases, the function increases in value between $x =$ and $x =$. and again between $x =$ and $x =$
3. As x increases, the function decreases in value between $x =$ . . and $x =$.
4. For which values of x is the gradient (slope) of the graph zero?
5. Solve the equation $2x^3 - 5x + 6 = 0$ Therefore, how many of its roots are said to be imaginary?
6. Solve $2x^3 - 5x + 6 = 2\ 96$ (approximately). Explain why in this case we may say that two of the roots are equal
7. Solve

(i) $2x^3 - 5x + 6 = 4.$ (ii) $2x^3 - 5x - 0\ 75 = 0$ (iii) $2x^3 = 5x + 2$

(iv) $2x^3 = 5x + 21\ 5$ (v) $4x^3 = 10x - 9$ (vi) $x^3 - 2\ 5x + 3 = 3.$

8. Consider the equation $2x^3 - 5x + 6 = k$

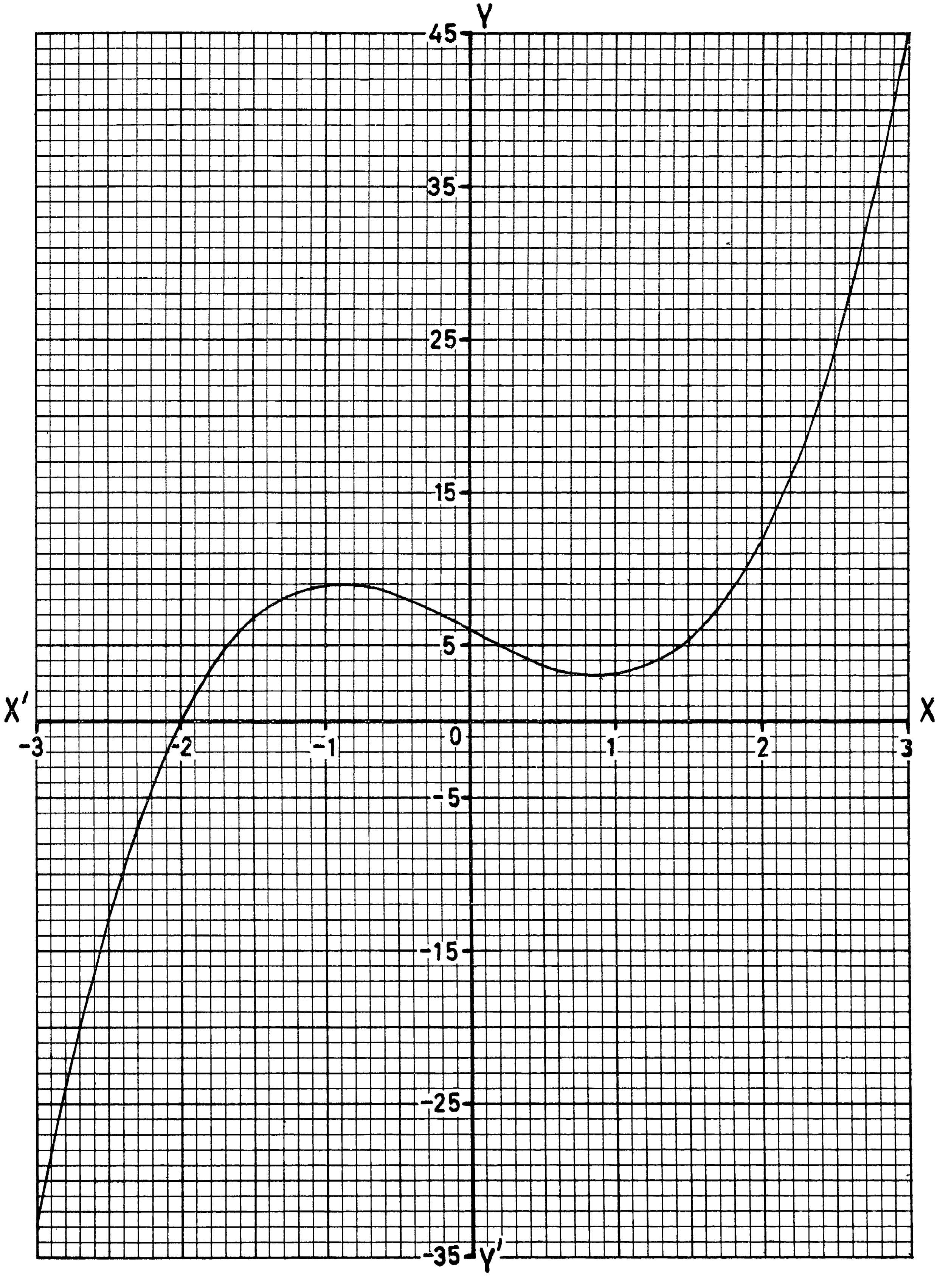
(i) How many roots has the equation if $k < 2\ 96$ or $k > 9\ 04$?

(ii) State the number and nature of the roots of the equation if $k = 2\ 96$ or $k = 9\ 04$

(iii) How many distinct roots has the equation if k lies between $2\ 96$ and $9\ 04$?

* On either side of a **point of inflection** the direction of slope of a curve is the same (if we move continuously from one end to the other) Compare this with a turning point where the slope is upward on one side and downward on the other

GRAPH OF $2x^3 - 5x + 6$



Example (3)

The curve opposite is the graph of $y = 6 + 7x - x^3$ from $x = -3\frac{1}{2}$ to $x = 3\frac{1}{2}$

Exercises

Use your graph to work the following

1. Within the limits shown state between which values of x the function $6 + 7x - x^3$

(*a*) Decreases as x increases
(*b*) Increases as x increases
(*c*) Is negative

2. Write down

(*a*) The maximum turning value of $6 + 7x - x^3$ and the corresponding value of x
(*b*) The minimum turning value of $6 + 7x - x^3$ and the corresponding value of x.

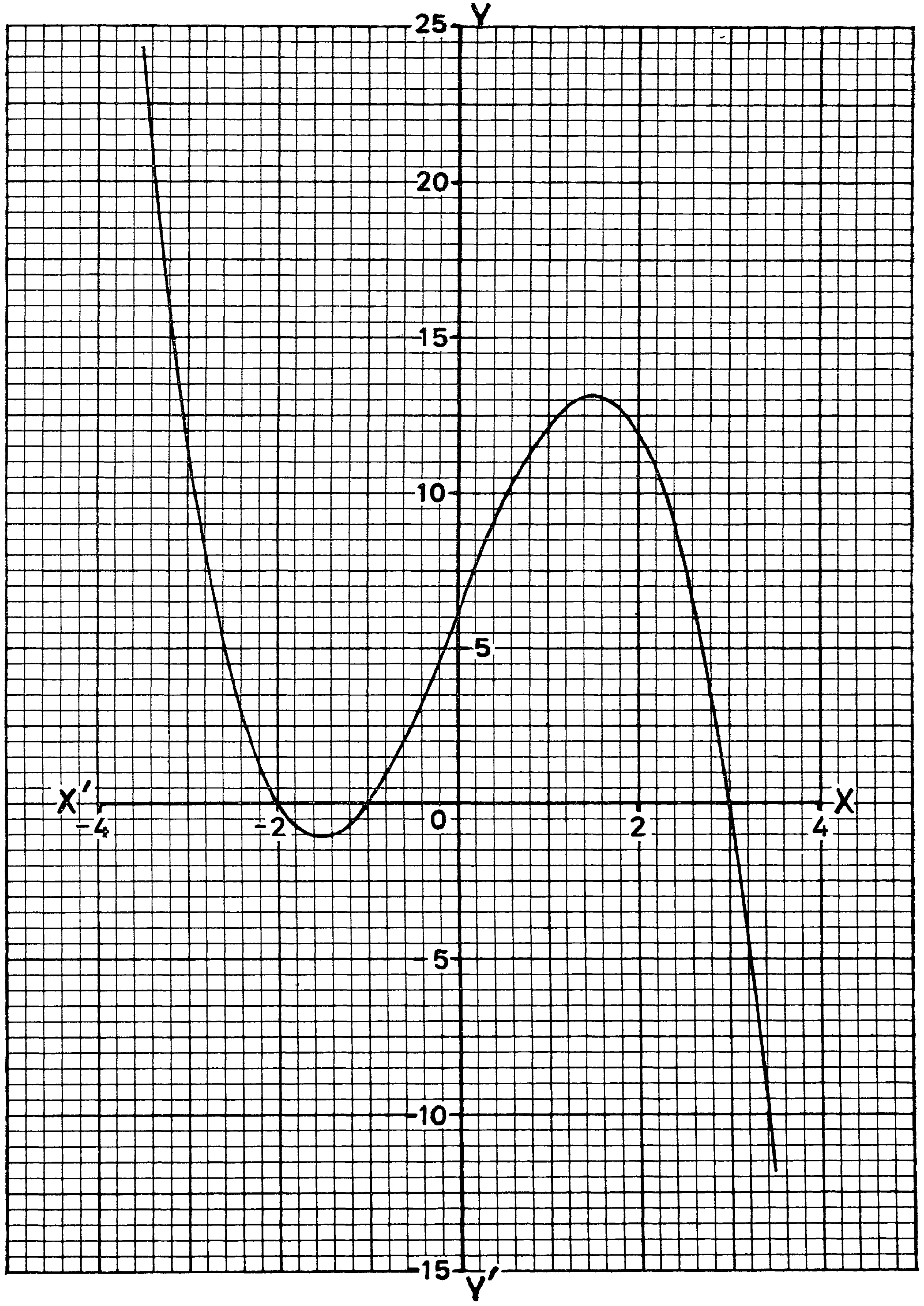
3. Solve, correct to two decimal places (where necessary).

(i) $6 + 7x - x^3 = 0$ (ii) $6 + 7x - x^3 = 3$
(iii) $6 + 7x - x^3 = -0\,5$. (iv) $7x - x^3 = 0$
(v) $6 + 7x - x^3 = -3$ (vi) $12 + 14x - 2x^3 = 21$

4. Consider the equation $6 + 7x - x^3 = k$

(*a*) State between which values of k the equation has three distinct roots
(*b*) State for which two values of k the equation has one distinct root and two equal roots Write down these roots in each case
(*c*) State for which values of k the equation has only one root

GRAPH OF $y = 6 + 7x - x^3$



Example (4)

The curved graph of $y = x(x + a)(x - b)$ is shown opposite for a limited range of values of x.

Exercises

1. By noting the values of x where the curve crosses the x-axis, write the values of a and b in the function $x(x + a)(x - b)$

2. State the range(s) of values of x for which the function is (i) positive, (ii) negative, (iii) decreasing as x increases

3. Write down the maximum and minimum turning values of the function and give the corresponding values of x correct to two decimal places

4. Solve, correct to two decimal places

(i) $x^3 + x^2 - 2x = -1$ (ii) $x^3 + x^2 - 2x = 3$

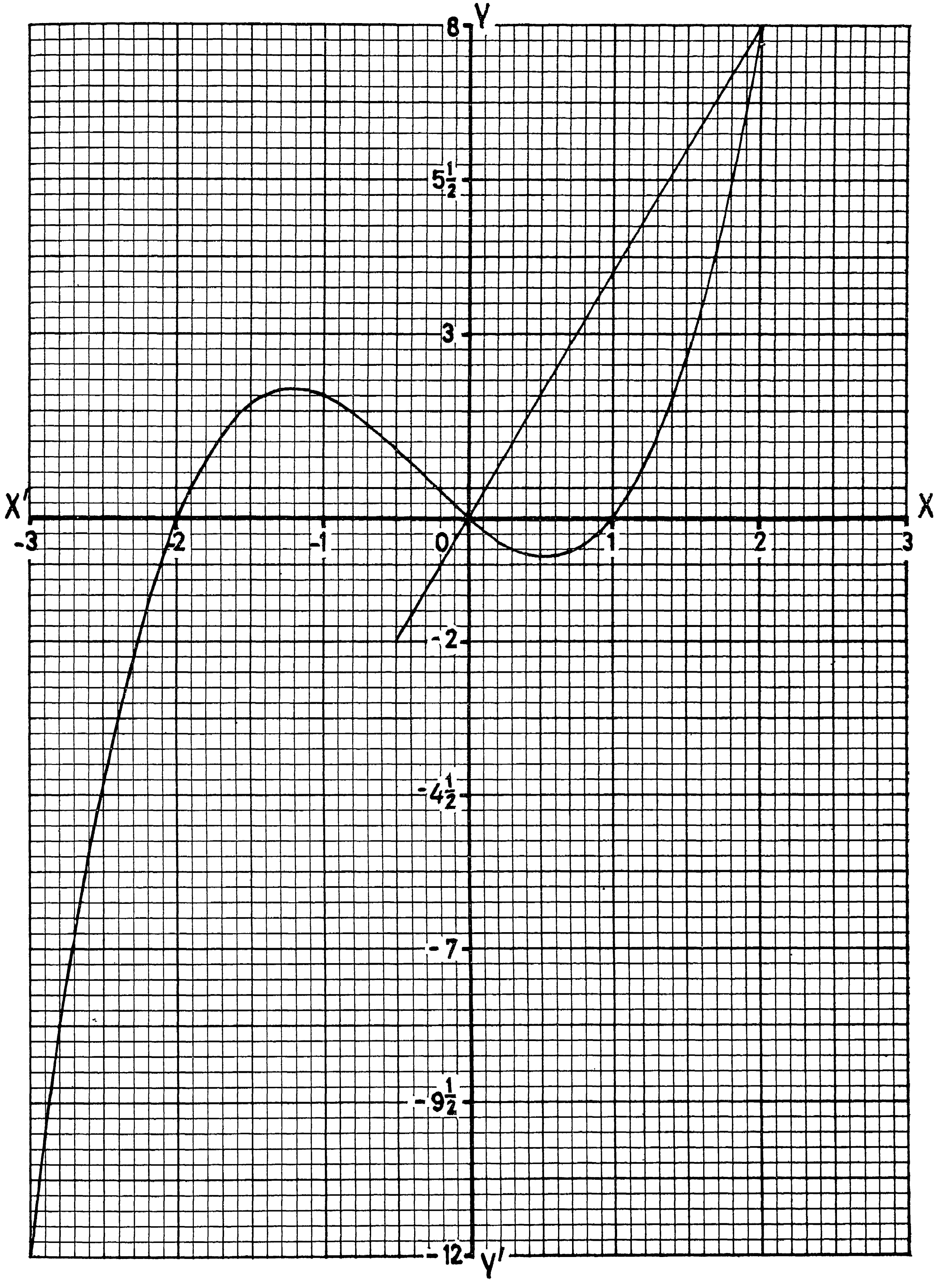
(iii) $2x^3 + 2x^2 - 4x = 1$ (iv) $4x^3 + 4x^2 - 8x = 5$

5. (*a*) State why the equation of the straight line shown is of the form $y = mx$ and find m

(*b*) The points where the straight line cuts the curve satisfy the equation . = .

(*c*) By factorizing, solve this equation and so find the co-ordinates of the third point in which the straight line should cut the curve

GRAPH OF $x(x+a)(x-b)$



Solution of Cubic Equations—Alternative Method

Example (5)

Solve graphically the equation $x^3 - 6x + 4 = 0$ Rewriting this equation in the form $x^3 = 6x - 4$ we see that the roots may be obtained by drawing the graphs of $y = x^3$ and $y = 6x - 4$ which are shown opposite

Exercises

1. Copy the graphs opposite (You may use the table on page 68 to plot the points for $y = x^3$)

2. From the points of intersection of $y = x^3$ and $y = 6x - 4$ write down, correct to two decimal places, three solutions of $x^3 - 6x + 4 = 0$ Compare your answers with those read from the graphs opposite

3. (*a*) By drawing the appropriate straight lines, find two equations of the forms

$$x^3 - 6x + k = 0$$

which have one distinct root and two equal roots.

(*b*) Write down the roots in each case, correct to two decimal places

4. Say how many roots $x^3 - 6x + k = 0$ will have if

(i) $k = +6$ (ii) $k = -6$ (iii) $k = +5$

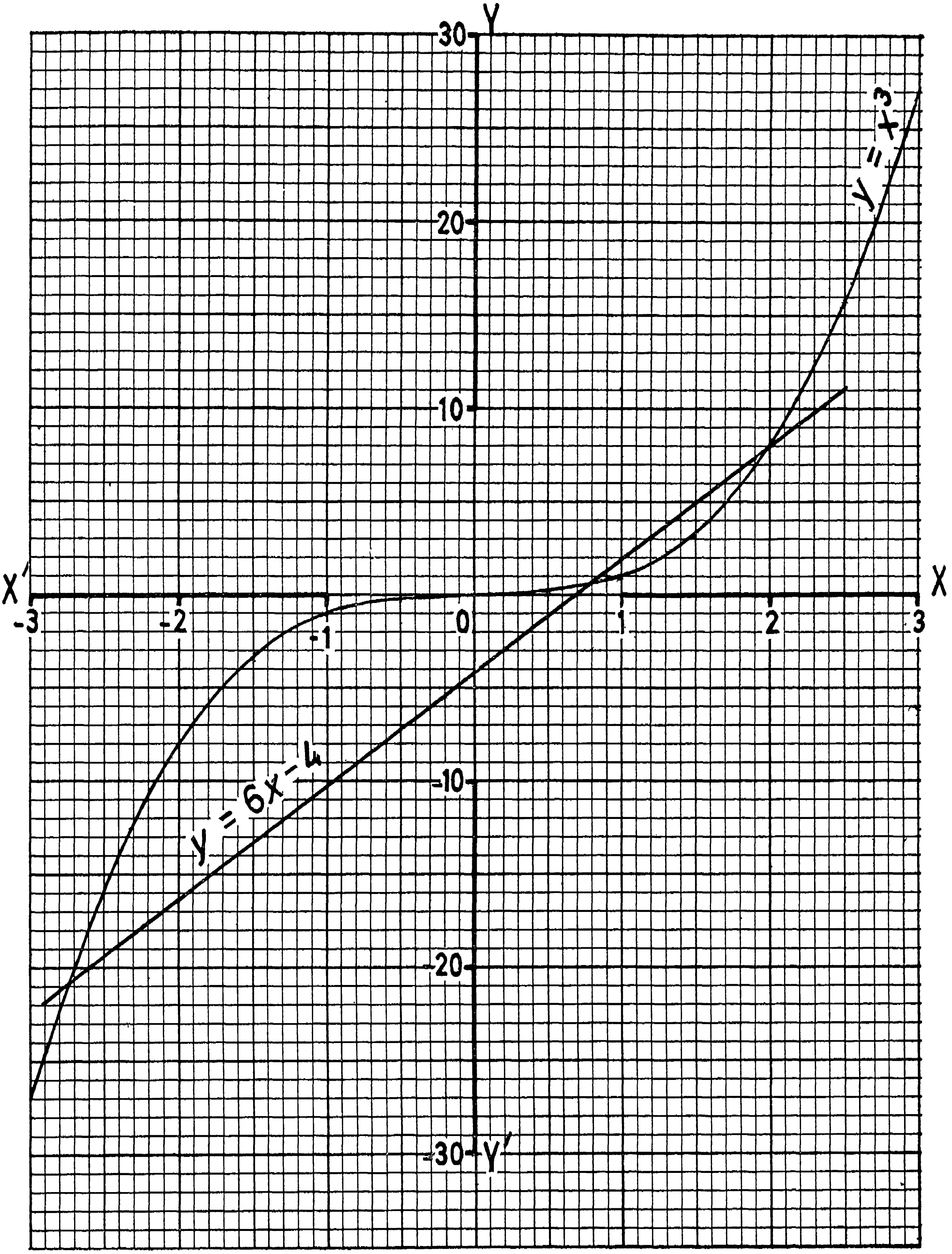
5. By drawing the appropriate straight line graphs, find, correct to two decimal places, the roots of

(i) $x^3 - 3x - 1 = 0$ (ii) $x^3 = 2x - 1$ (iii) $x^3 - 5x - 7 = 0$

(iv) $x^3 + 3x - 1 = 0$ (v) $x^3 = 1 - 2x$ (vi) $x^3 + 5x + 7 = 0$.

6. Say how many solutions can be found for $x^3 + lx + k = 0$ where l is positive and k may be either positive or negative

GRAPHS OF $y = x^3$, $y = 6x - 4$



B The Equation $y = \frac{k}{x}$ (The Hyperbola)

Example (6)

When a distance of 60 miles is covered in varying times x hours, the average speed y miles per hour is given by $y = \frac{60}{x}$, or $xy = 60$ y is inversely proportional to x because when x is doubled, y is halved, and when x is multiplied by three, y is divided by three, and so on

The curve corresponding to two quantities which are in inverse proportion is called a **hyperbola.** Taken from this problem x and y will both be positive and the curve will lie in the first quadrant only But for algebraic purposes negative values of x have also been used in drawing the curve opposite

Exercises

1. Complete the table used in drawing the part of the graph of $y = \frac{60}{x}$ for positive values of x.

x	0 5	1	2	3	4	5	6	10	12
$y = \frac{60}{x}$			30						5

2. (*a*) Write down the values of y corresponding to x = 0 1, 0 01, 0 001, 0 0000001
(*b*) Use these results to complete the statement
"As x decreases from 0 1 to 0 0000001 y from to . and the graph moves closer and closer to the axis of "

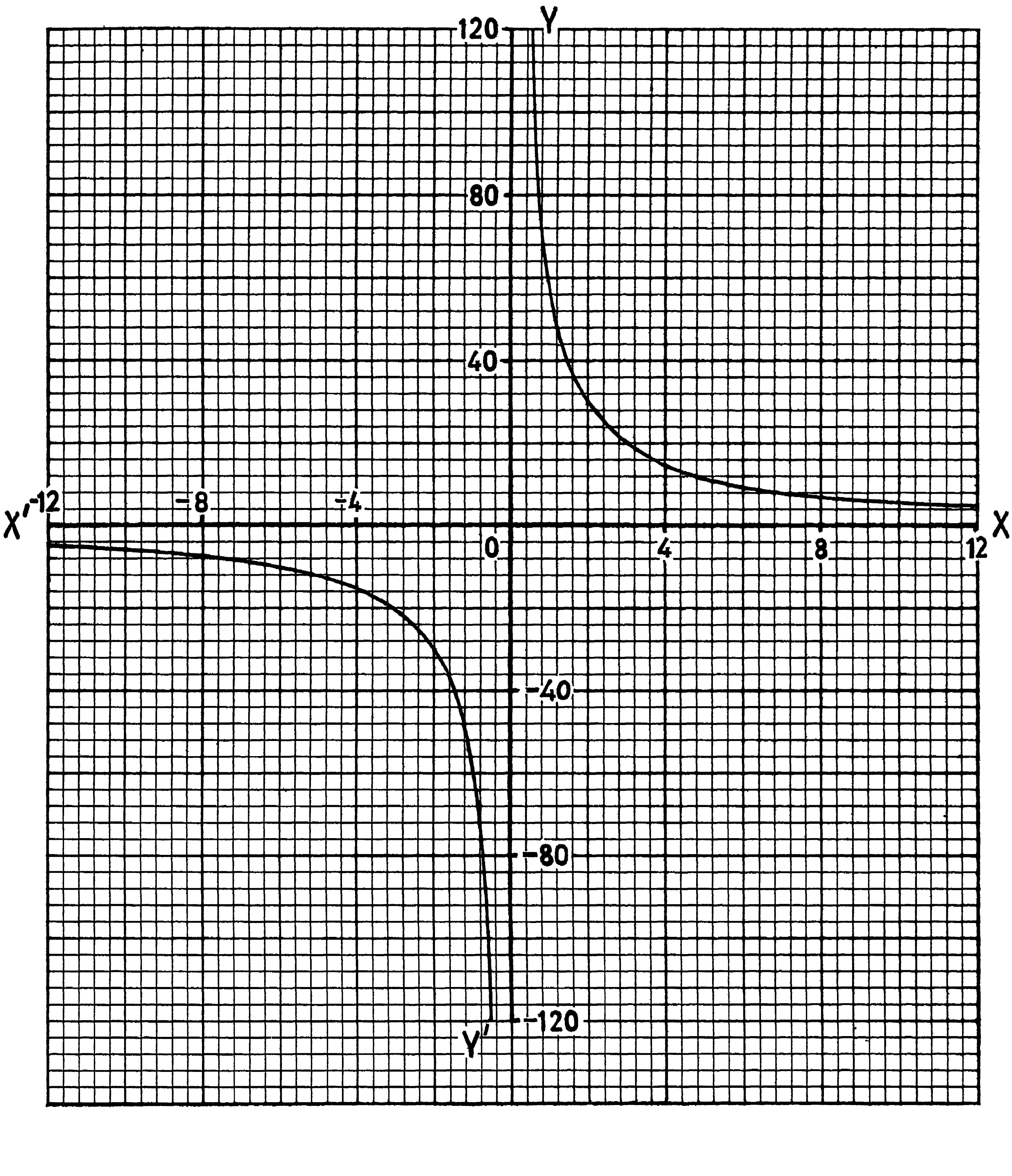
3. (*a*) Write down the values of y corresponding to x = 100, 1000, 10,000, 10,000,000
(*b*) Now complete the statement
"As x increases from 10 to 10,000,000 y from to and the graph moves closer and closer to the axis of . . "

4. (*a*) From the original problem say whether there is a time corresponding to a speed of 0 miles per hour
(*b*) Say if any value can be given to y in the equation $y = \frac{60}{x}$ when $x = 0$
(*c*) Say also if there is any value of x which will make $y = 0$
(*d*) State whether the curve will ever meet either the y-axis as x is positive and decreasing, or the x-axis as x is positive and increasing.

5. (*a*) Rewrite the table in Exercise **1**, substituting negative values of x for the positive values given
(*b*) Explain the effect of these negative values on the position and shape of this part of the curve

6. Plot the points given in your two tables of values and compare your completed graphs with those opposite

GRAPH OF $y = \frac{60}{x}$



7. By drawing the appropriate straight line graphs, solve (graphically)

(i) $xy = 60$, $y = 2x + 2$ (ii) $xy = 60$, $y = 3x - 8$ (iii) $xy = 60$, $y = -8x + 68$

8. Without drawing the curve, describe the graph of $y = -\frac{60}{x}$

Asymptotes

We saw that the graph of $y = \frac{60}{x}$ approached closer and closer to the x-axis (that is, $y = 0$) as x became larger and larger, and to the y-axis (that is, $x = 0$) as y became larger and larger For this graph the x-axis and the y-axis are called **asymptotes,** lines which the graph approaches but never meets.

(1) Consider $y = \frac{12}{x + 2}$ y becomes very large as x approaches -2

Rewriting the equation in the form $x + 2 = \frac{12}{y}$, x becomes very large as y approaches 0

Therefore, for the graph of $y = \frac{12}{x + 2}$ the asymptotes are the lines $x = -2$ and $y = 0$

(2) Consider $y = \frac{12}{x} - 3$ y becomes very large as x approaches 0

Rewriting the equation firstly as

$$y + 3 = \frac{12}{x}$$

and then

$$x = \frac{12}{y + 3}$$

x becomes very large as y approaches -3 Therefore, for the graph of $y = \frac{12}{x} -3$ the asymptotes are the lines $x = 0$ and $y = -3$

Exercises on Chapter 7

1. Draw the graph of $x^3 - 2x$ between $x = -2$ and $x = 2$, using 1 inch to 1 unit for x and 1 inch to 2 units for y
Use your graph to solve, correct to two decimal places, $x^3 - 2x - 1 = 0$
Without actually drawing the graphs, explain briefly another method of solving this equation graphically

2. Draw the graph of $2x^3 - 10x - 5 = 0$ using values of x between $x = -3$ and $x = 3$
Use your graph to solve, correct to two decimal places, the equations

(i) $2x^3 - 10x - 5 = 0$ (ii) $2x^3 - 10x - 5 = 2$ (iii) $2x^3 - 10x - 5 = -3$
(iv) $2x^3 - 10x = -4$. (v) $2x^3 = 10x + 13$ (vi) $x^3 - 5x + 1 = 0$

3. Draw the graph of the function $\dfrac{x(x^2-4)}{2}$ from $x = -3$ to $x = 3$. From your graph find

(i) The value of x for which the function is equal to 6.
(ii) The x-co-ordinate of the points on the curve which are equidistant from $(-2, 10)$ and $(2, -5)$.

4. The table gives the values of $y = 2x^3 - 6x + 3$ from $x = -2\frac{1}{2}$ to $x = 2\frac{1}{2}$.

x	$-2\frac{1}{2}$	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$
y	$-13\frac{1}{4}$	-1	$5\frac{1}{4}$	7	$5\frac{3}{4}$	3	$\frac{1}{4}$	-1	$\frac{3}{4}$	7	$19\frac{1}{4}$

Using 1 inch to 1 unit for x and 1 inch to 5 units for y, draw the graph of

$$2x^3 - 6x + 3$$

From your graph find·

(a) Between which values of x (i) the function is increasing as x increases, (ii) the function is negative
(b) The turning values of the function and the corresponding values of x.
(c) The solution of the equation $2x^3 - 6x + 3 = 0$

5. Draw the graph of $y = x^3 - 15x - 1$ from $x = -4$ to $x = 4$, using the values in the following table

x	-4	-3	-2	-1	0	1	2	3	4
y	-5	17	21	13	-1	-15	-23	-19	3

Use 1 inch to 1 unit for x and 1 inch to 10 units for y From your graph determine

(a) Between which negative values of x the function $x^3 - 15x - 1$ is positive.
(b) The maximum and minimum turning values of $x^3 - 15x - 1$.
(c) The intercept of the graph on the y-axis
(d) The roots of the equations

(i) $x^3 - 15x - 1 = 0$. (ii) $x^3 - 15x = 3$. (iii) $x^3 - 15x + 2 = 0$.

6. Plot on a scale of 1 inch to 1 unit for x and 1 inch to 2 units for y, the graph of

$$y = \tfrac{1}{2}(x^3 - 8x + 3)$$

and from your graph read off the roots of

(i) $x^3 - 8x + 3 = 0$. (ii) $x^3 - 8x + 3 = 5$ (iii) $x^3 - 8x + 4 = 0$.

Now draw in the appropriate straight line graph and solve.

(iv) $x^3 - 8x + 3 = -x$ That is, $x^3 - 7x + 3 = 0$

7. Draw the graph of $(3 - 2x)(x + 1)^2$ from $x = -3$ to $x = 2$, using 1 inch to 10 units for y.

(a) Reading values of x to the second decimal place and values of y to the first decimal place, find

(i) The x-co-ordinates of the maximum and minimum turning values of the function $(3 - 2x)(x + 1)^2$.
(ii) The values of x for which $(3 - 2x)(x + 1)^2$ is equal to 2.
(iii) The range of values of p for which the equation $(3 - 2x)(x + 1)^2 = p$ has three real roots.

(b) By drawing the appropriate straight line graph solve $(3 - 2x)(x + 1)^2 = -x$

(c) Two roots of the equation $(3 - 2x)(x + 1)^2 = ax + b$ are -2 and $-0 3$ Draw a straight line on the graph and so find the third root

8. Draw the graph of $y = 4x(4 - x)^2$ between $x = 0$ and $x = 5$, taking 1 inch to 1 unit for x and $\frac{1}{10}$ inch to 1 unit for y.

A square sheet of cardboard has an 8-inch side Out of each corner a square of side x inches is cut and the flaps which remain are turned up so as to form an open rectangular box of depth x inches Show that the volume of the box is $4x(4 - x)^2$ cubic inches Use your graph to find what size of square should be cut from each corner to form a box of maximum volume

9. Draw the graph of $y = \dfrac{12}{x}$ for values of x from $x = \pm\frac{1}{2}$ to $x = \pm 24$ (omitting the value $x = 0$), and a scale of 1 inch to 10 units on each axis

10. (a) Draw the graph of $y = \dfrac{12}{x + 1}$ for the following ranges of values of x·

(i) $-\frac{1}{2}$ to $+23$. (ii) -25 to $-1\frac{1}{2}$

Write down the equations of the asymptotes

(b) On the same graph sheet draw the graph of $y = \dfrac{12}{x - 1}$. Write down the equations of the asymptotes

11. Use the graphs of $y = \dfrac{6}{x}$ (for positive values of x) and $3x + 4y = 18$ to solve the equation

$$\frac{18 - 3x}{4} = \frac{6}{x}.$$

Check your solution algebraically.

12. (a) Rectangular metal strips of length x inches and breadth y inches are of different shapes but the same area, 36 square inches Write down the equation connecting x and y and draw the corresponding graph

(b) From the intersections of this graph and the appropriate straight line graph find which of these strips has a perimeter of 26 inches

Chapter 8

STATISTICS (1)

Statistics deals with the collection of facts and figures, their presentation in graphical or tabular form, and the measures which enable us to draw reasonable conclusions from the data so presented

Pictographs

Perhaps the most arresting way of illustrating statistics is in the form of pictures called **pictographs.** These may be of motor cars, houses, aeroplanes, for example, reduced in size, but drawn to a definite scale for purposes of comparison.

Fig. 9 illustrates

(*a*) How the deaths in road accidents among boys and among girls vary from age group to age group over the age range 1 to 13 years.

(*b*) How the number of deaths among boys in any of these groups compares with the number of deaths among girls

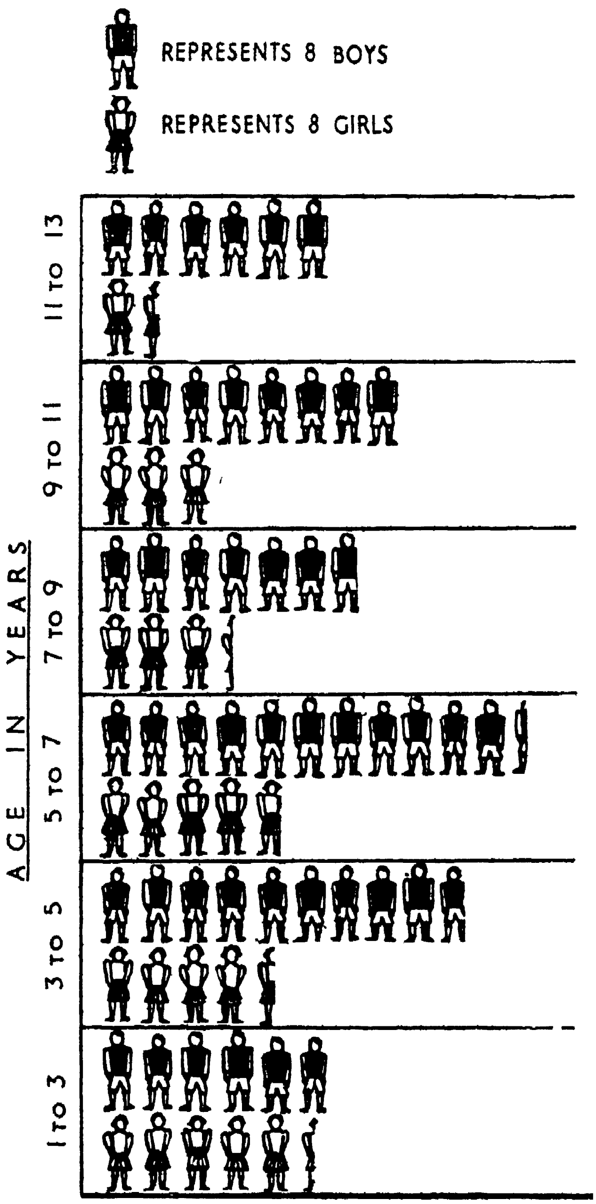


Fig 9

Bar Graphs

These graphs make comparisons by columns or bars of various heights or lengths They have two advantages over pictographs·

(*a*) They are easy to draw.
(*b*) They can be drawn and read to a greater degree of accuracy.

The accuracy depends, of course, on the size and range of the numbers involved

Fig 10 makes a comparison between some of the world's largest companies on an employment basis

Line Graphs

In illustrating statistics where not only rates of change but also trends are important the line graph is usually used Sales, employment, import and export figures, investments, bank deposits, and weekly wage rates are best illustrated by line graphs

Fig 11 gives a comparison between the amount of capital the United Kingdom, the United States, and other countries have invested in the Argentine over the period from 1930 to 1959

Limitations

It is important to understand the limitations of various statistical methods and to avoid drawing erroneous conclusions from the data or pictorial diagrams For example, where a bar diagram shows that the accidents in a particular area for the first three months of the year showed a substantial rise in 1959, the cause may not be increased traffic or increased speed, but especially icy conditions for that particular year. See Fig 12 opposite

Or, again, if a pictograph shows that 45,000 houses were being built per month in Western Germany in 1954 compared with approximately 30,000 houses per month in the United Kingdom, it cannot be assumed that the value of the West German effort is $1\frac{1}{2}$ times as great as the British Population, present housing shortage, labour force, and many other factors have to be considered

EMPLOYEES IN EIGHT COMPANIES (1960)

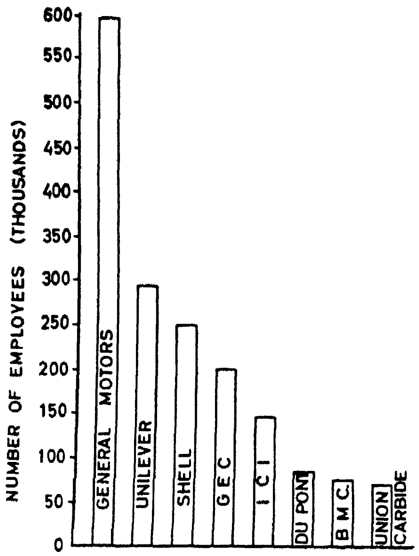


Fig 10

FOREIGN INVESTMENTS IN ARGENTINA

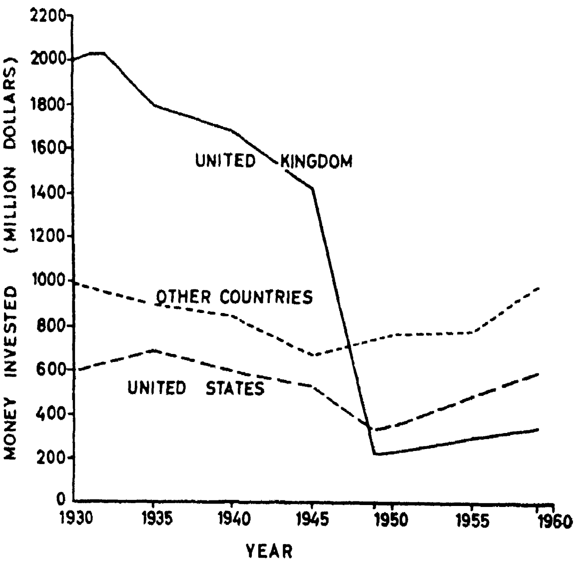


Fig 11

When reading graphs, care must be taken to note not only the scale but also the lowest value especially on the vertical axis. Consider the line graphs, Fig 13 and Fig. 14.

By comparing Fig. 14, where the lowest value on the vertical axis is 3200 and the scale is relatively large, with Fig 13 where the vertical scale has a true zero and the scale is small, we see how the false zero creates an impression of much greater progress

On the other hand, a patient's temperature is normally within the range 96°F–104°F and to give a clear indication of its fluctuation over such a small range the vertical scale must begin around 94°F and not at 0°F.

Greater misconceptions are caused when the scale is omitted altogether A graph must have a scale on each axis before it has any meaning at all

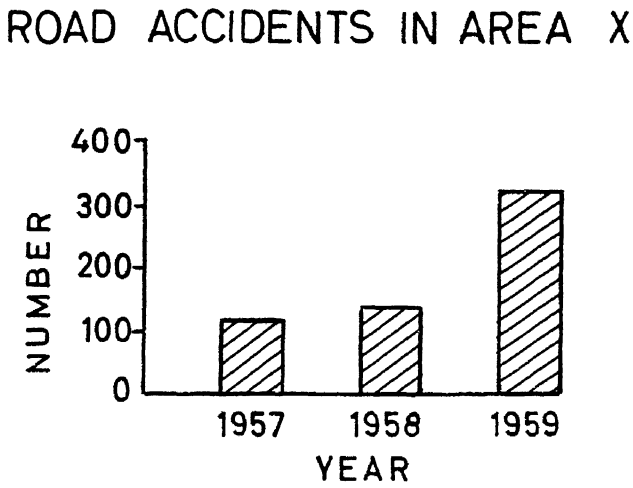


Fig 12

Fig 13

Fig 14

Pie Charts

Pie charts (or circle diagrams) are used to show the various parts into which a whole is divided—a circle divided into a number of sectors of various sizes. The fraction of the whole allotted to each item is equal to the fraction of 360° which the sector showing that item subtends at the centre of the circle

Example (1)

Fig. 15 shows the allocation of time each week to various subjects for pupils following a Science course This covers 40 school periods per week The Science sector subtends a 90° angle at the centre of the circle and as $90°/360° = \frac{1}{4}$, the number of Science periods per week is $\frac{1}{4}$ of $40 = 10$

Exercise

Measure all other angles at the centre of the circle and so calculate how many periods per week are allocated to each of the other subjects shown

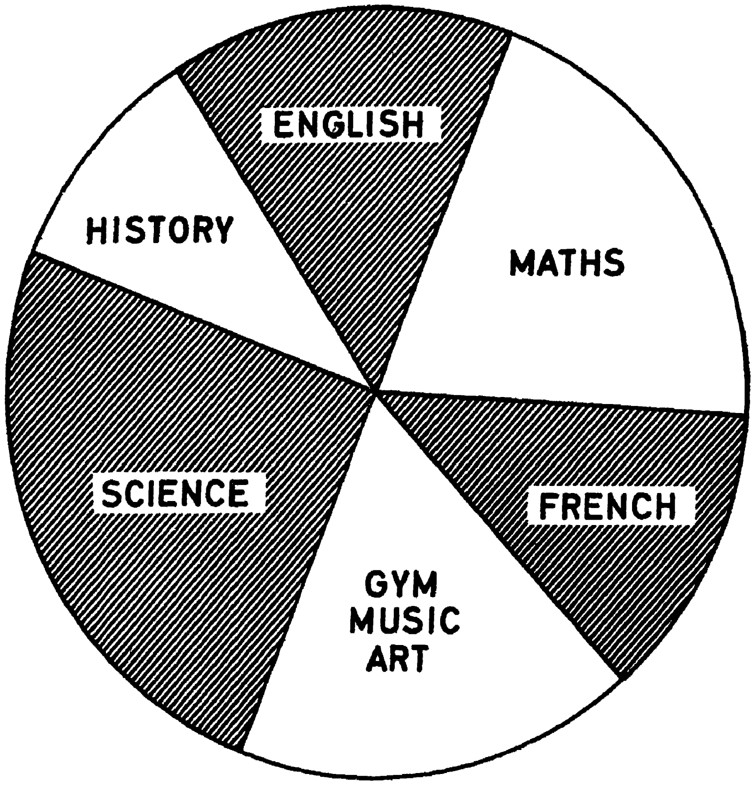


Fig 15

ROAD DEATHS (1956) IN AGE GROUP 1 TO 19

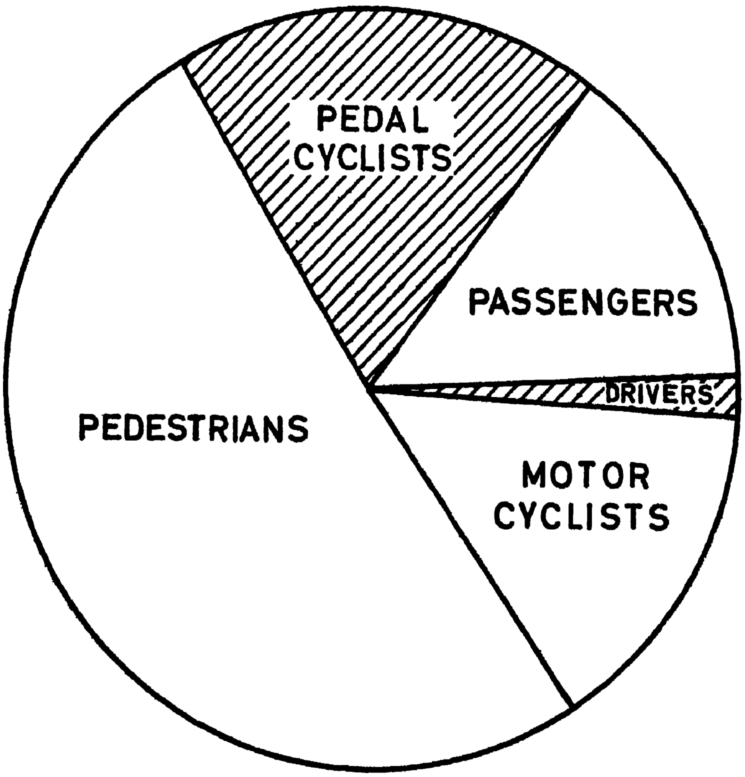


Fig 16

Example (2)

By expressing the sector angles as fractions of 360°, the parts making up the whole circle can easily be converted arithmetically to percentages. Should the number of sectors be large, a simple conversion graph can be used because a high degree of accuracy is usually unnecessary

Exercises

Fig 16 shows how the road deaths in 1956 for the age group 1 to 19 were made up

1. Measure and tabulate each sector angle

2. Copy the conversion graph (Fig. 17) on 1-inch graph paper

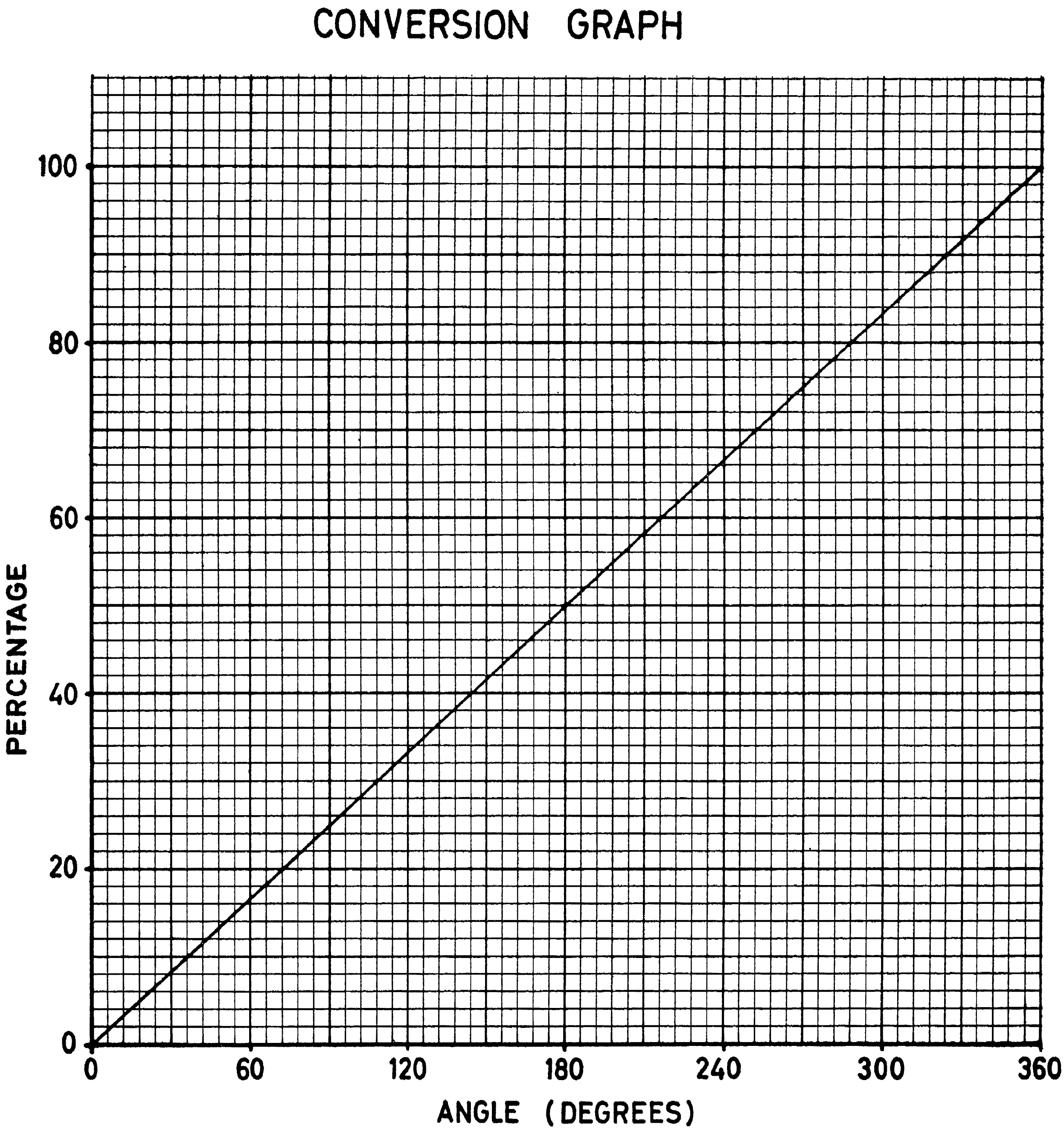


FIG 17

3. Use the conversion graph to express the deaths in the various groups as percentages of the total casualties within the age range 1 to 19.
4. By calculation, check your percentages for: (*a*) motor cyclists, and (*b*) pedal cyclists.

Example (3)

Figs 18 and 19 show the division of Mr Dow's income per month during the years 1960 and 1961 The areas of the circles are in proportion to his monthly income for these years. In 1960 Mr Dow was a Sales Representative earning £100 per month, and in 1961 he was Sales Manager.

Exercises

1. Measure the radii of the two circles and calculate the ratio of their areas
2. Hence calculate Mr Dow's income per month in 1961
3. Compare the monthly cash amounts allocated to clothes in 1960 and in 1961
4. Say which items have equal shares of the income (*a*) in 1960 and (*b*) in 1961
5. (*a*) Which items represent a smaller percentage of the monthly income in 1961 than in 1960?
   (*b*) By comparing the cash amounts for these items, show that they are all greater in 1961 than in 1960
6. Calculate the sector angle for housekeeping in (*a*) 1960 and (*b*) 1961
7. If one-third of the increase in expenditure on rent and rates is caused by increased rating, how much more does Mr Dow pay for rates in 1961?

Note It will now be realized that pie charts have limited value owing to:

(1) the difficulty of gauging and comparing the sizes of the sectors without calculation, measurement, or percentage figures in the sectors,
(2) the lack of accuracy

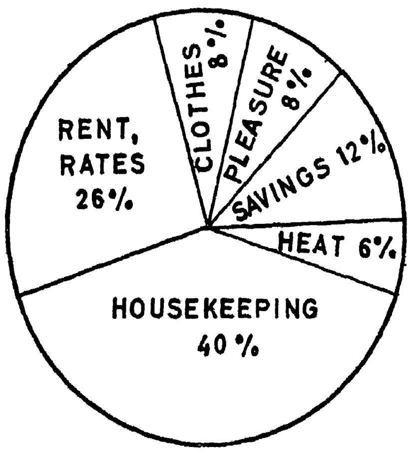


Fig 18

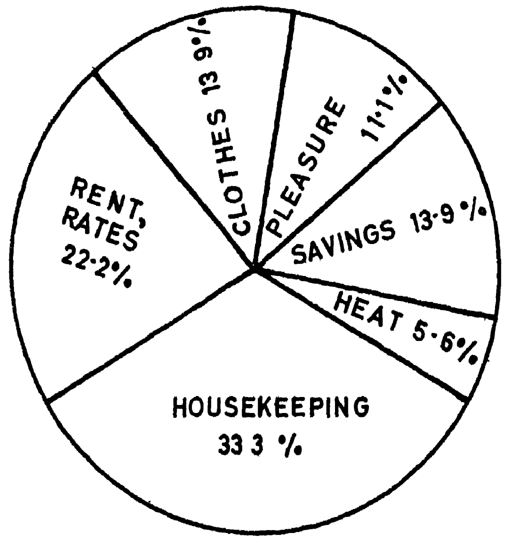


Fig 19

Exercises on Chapter 8

1. Each £1 of sales revenue of a firm is accounted for as follows: wages, 7s.; materials, 4s. 2d.; expenses, 2s. 6d.; taxation, 4s. 6d.; profits, 1s. 10d.
Using a circle of radius 2 inches, draw a pie chart to represent this division.

2. In 1957 the supply of electricity to consumers in a certain area was divided as follows: domestic, 30%, commercial, 15%, industrial, 51%, farm and public lighting, 4%. Draw a circle of radius 1·8 inches, and, by *calculating* the necessary sector angles, draw a circle diagram to illustrate these percentages.

3. In Scotland in 1957 the principal classes of consumers and the corresponding percentages of total gas supplied were:

Consumer	*Percentage*
Domestic prepayment	40·3
Domestic credit	18·9
Industrial	21
Commercial	14·3
Public administration, lighting, etc.	5·1

(*a*) Use a conversion graph (see p. 87) to help you to show these divisions on a pie chart of radius 1·8 inches.
(*b*) Illustrate the same figures by a bar graph.
(*c*) Discuss the merits of the two graphs with regard to:

(i) Ease of drawing.
(ii) Presentation of facts individually.
(iii) Presentation of facts as part of a whole.
(iv) Accuracy.

4. The pictograph (Fig. 20) is used to illustrate that the ratio of sheet steel production for a year in three areas of the United Kingdom was 4·5 : 3 : 1.

(*a*) Check that each of the dimensions of length, breadth, and thickness measured in millimetres is approximately in this ratio.
(*b*) State, with reasons, why you consider that the pictograph does not give a clear picture of this relationship.
(*c*) Using the diagram for Area C as 1 unit, draw a *sketch* which gives a more accurate comparison of production in the three areas.

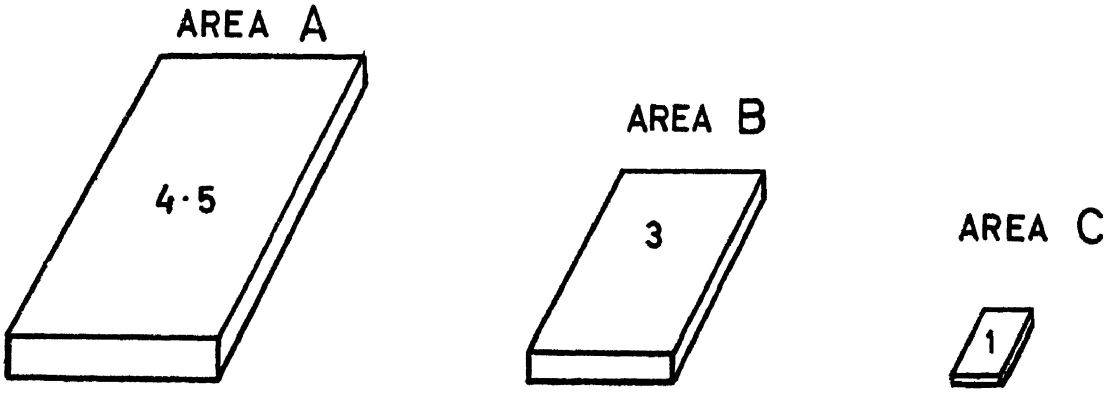


Fig. 20

5. The pie chart (Fig 21) shows the divisions of the U K chemical industry in terms of 1948 capacity.

(a) Measure the sector angle for (i) pigments and (ii) heavy inorganic chemicals Say which other branches are respectively equal to these two in approximate size

(b) Find what percentage of the total is devoted to (i) Gases (ii) Miscellaneous (iii) Dyestuffs and Intermediates (by calculation)

(c) Say which four branches make up approximately half the industry

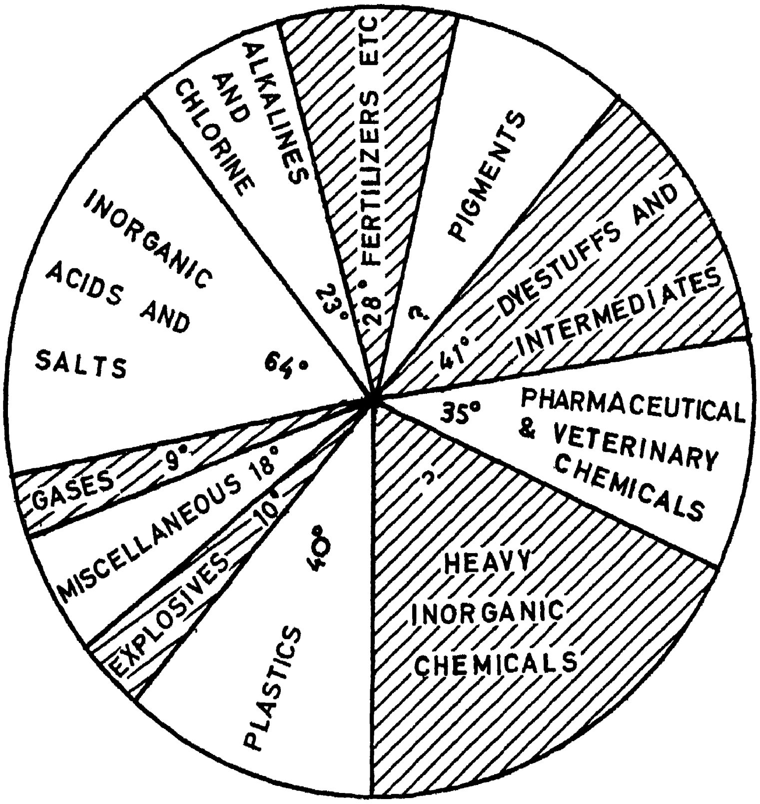


FIG 21

Chapter 9

STATISTICS (2)

Frequency

A Frequency Distribution

To study the performance of a group of 100 children in Arithmetic each one is given the same 10 sums to work The results are then tabulated to determine the number of times the examination papers show 10 sums correct, 9 sums correct, and so on. This number of times is called the frequency and the table is called a frequency distribution

No. of Sums Correct		*Frequency*
10	11	2
9	$\cancel{1111}$	5
8	$\cancel{1111}$ 1111	9
7	$\cancel{1111}$ $\cancel{1111}$ $\cancel{1111}$	15
6	$\cancel{1111}$ $\cancel{1111}$ $\cancel{1111}$ $\cancel{1111}$ $\cancel{1111}$	25
5	$\cancel{1111}$ $\cancel{1111}$ $\cancel{1111}$ 111	18
4	$\cancel{1111}$ $\cancel{1111}$ 1111	14
3	$\cancel{1111}$ 1	6
2	111	3
1	11	2
0	1	1

The method of counting numbers in each group is to make one tally-mark in the appropriate row in the centre of the table each time an examination paper is counted. Counting is done by fives with every fifth tally-mark being made diagonally across the preceding four

The Mode

The number of sums correct which occurs most frequently is 6. This is called the **mode** and is an indication of the character of the group The mode is what many people mean when they speak of the man of average income, or the average golfer, that is, the one which is characteristic of the group because it occurs most frequently

The Histogram

This is a diagram used to illustrate a frequency distribution It shows the pattern of the distribution, whether, for example, it is symmetrical about a central value Rectangles, based on the horizontal axis, are drawn with their areas proportional to the various frequencies If the breadths of the rectangles are equal, their heights will be proportional to the frequencies

Fig. 22 shows the histogram for the Arithmetic test already tabulated.

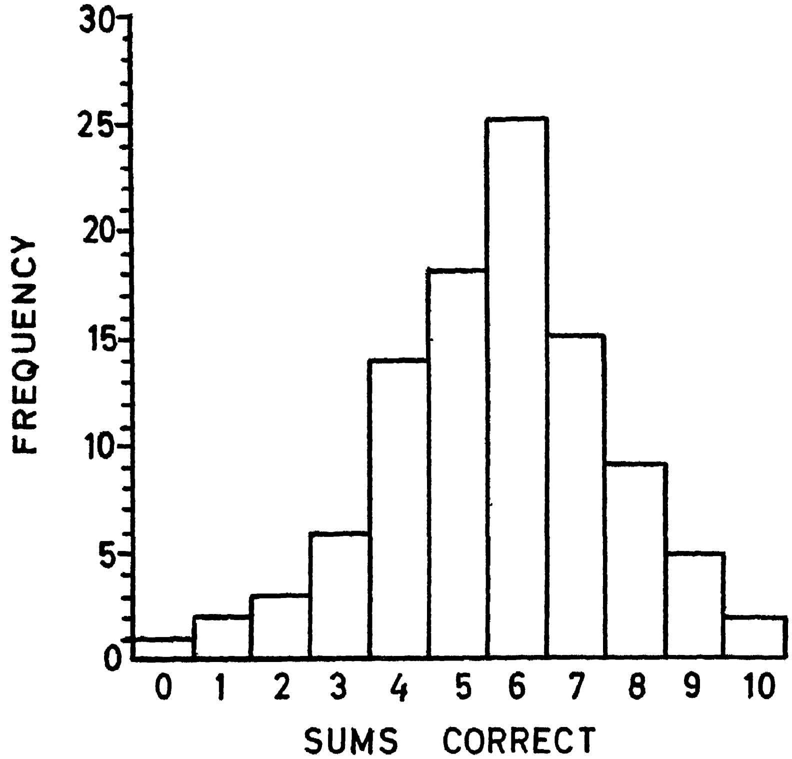


Fig 22

The histogram (Fig 23 opposite) shows a shop's sales of men's shoes of various sizes for one year.

Exercises

1. Write down the number of pairs sold in each of the sizes and so find the total sales
2. Which is the modal size?
3. (*a*) Find the average number of pairs sold per size (to the nearest whole number)
(*b*) Explain why this average is of no value in considering future stocks
4. (*a*) If orders of new stock were made quarterly, suggest how many pairs of each of the four most popular sizes you would order for the next three months
(*b*) Indicate factors besides those shown which might influence your order

Frequency Distributions and Class Intervals

When pupils in a country school are grouped according to age as shown in the table, the intervals between the lowest and highest ages of each group are called **class intervals.** The class interval here is two years

Age in Years	*No of Pupils*
5 and under 7	9
7 and under 9	18
9 and under 11	20
11 and under 13	14
13 and under 15	4

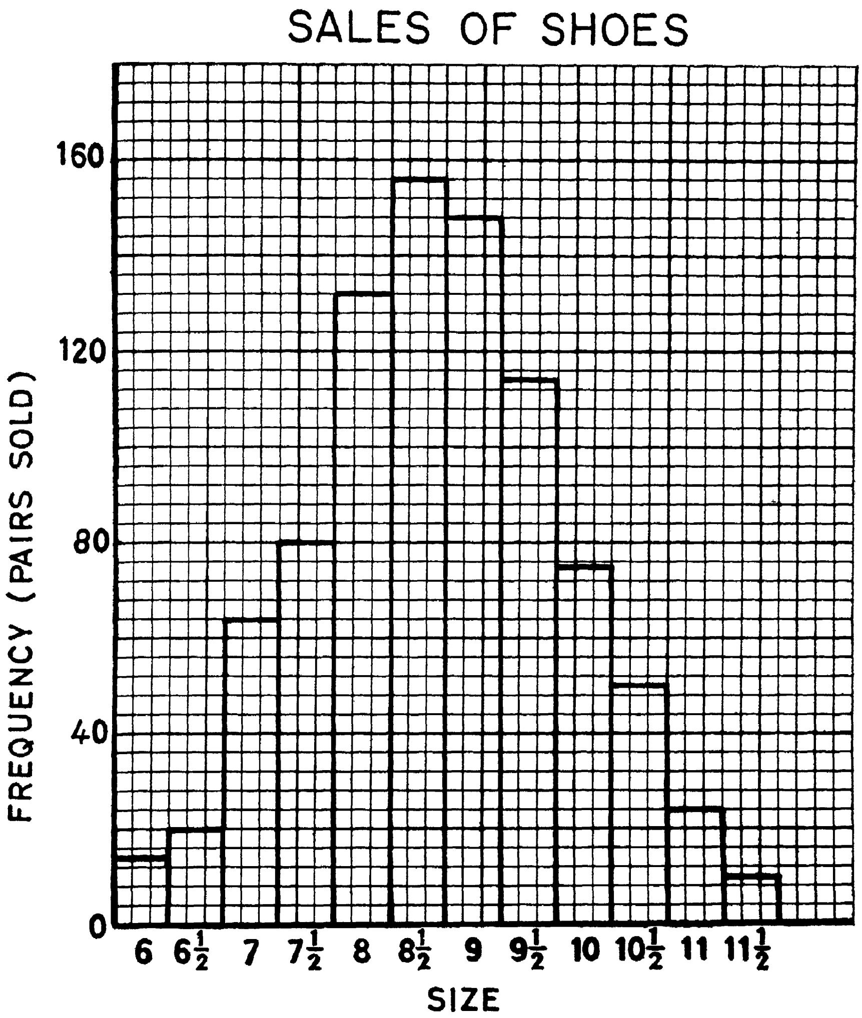


Fig 23

The number of class intervals should be between 7 and 15, and the range for each interval should normally be either 5 or 10 units. Where the range is small the best interval is 3 units.

Example A Class Intervals of 5 Units

Heights of Boys (inches)	*No of Boys* (Frequency)
45–49	7
50–54	63
55–59	98
60–64	135
65–69	82
70–74	14
75–79	1
Total	400

Example B Class Intervals of 10 Units

Average Day Temperature (°F)	*No. of Days* (Frequency)
20–29	2
30–39	47
40–49	93
50–59	129
60–69	70
70–79	15
80–89	5
90–99	4
Total	365

In Example A the interval 50–54 includes all heights from $49\frac{1}{2}$ inches to just under $54\frac{1}{2}$ inches, assuming that heights are recorded to the nearest inch The **mid-interval value** is 52 inches

In Example B, where all figures are degrees Fahrenheit, the interval 40–49 gives mathematical limits of $39\frac{1}{2}$ to $49\frac{1}{2}$ and a mid-interval value of $44\frac{1}{2}$

In examples on ages, mathematical limits do not apply
Age group 5–14 should be treated as 5 and under 15
Age group 15–24 should be treated as 15 and under 25
Age group 25–34 should be treated as 25 and under 35

Mid-interval values for these groups are 10, 20, and 30 respectively.

The histogram corresponding to Example A is shown in Fig 24.

Modal Class

Where class intervals take the place of individual values, the class which shows the greatest frequency is called the **modal class.** In Example A this is 60–64 inches In our example on the Arithmetic test the mode was the single value "6 sums correct," and this value actually appeared most frequently In Example A there may be no actual height which occurs most frequently, as no two boys may be exactly the same height

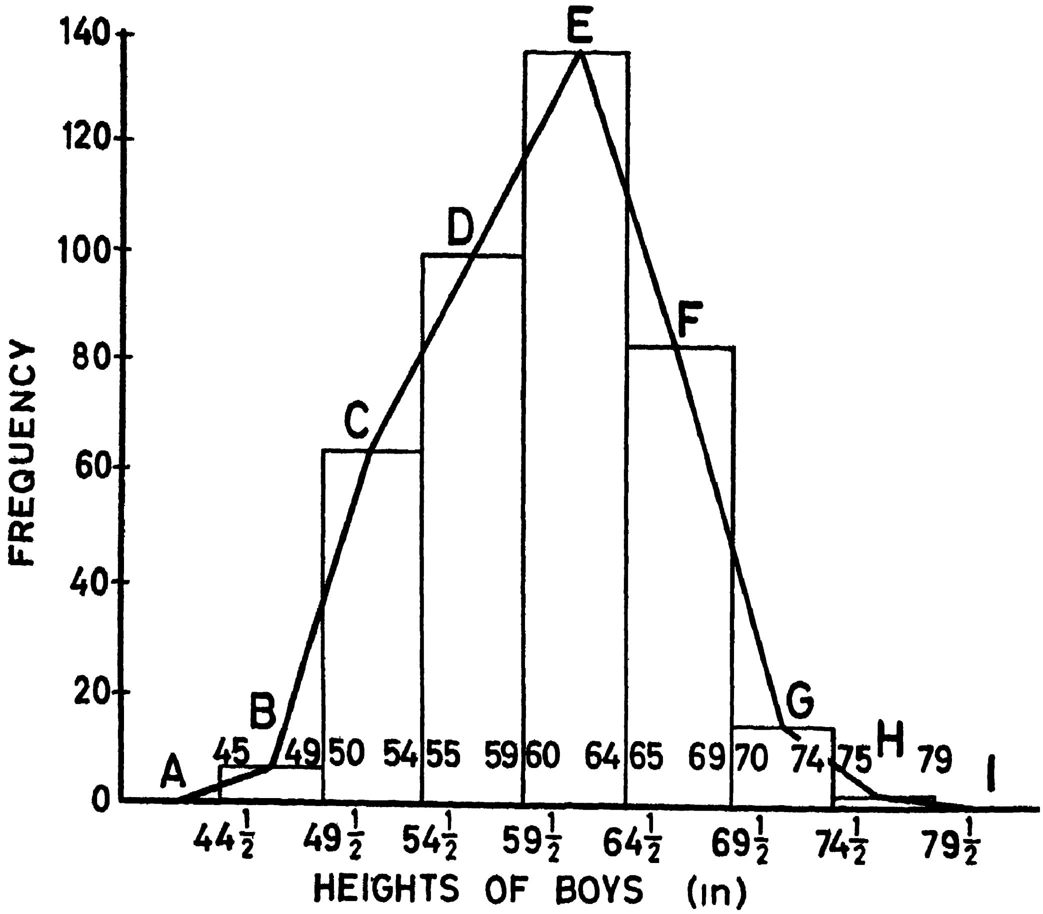


Fig 24

The Frequency Polygon

By joining the mid-points of the tops of the rectangles of a histogram, we form a frequency polygon ABCDEFGHI as shown in Fig 24. Note the way in which the lines are continued beyond the first and last rectangles. This ensures that the polygon will have the same area as the histogram if the class intervals are all the same (Is there a geometrical reason for the equality of these areas?) As two or more frequency polygons can be drawn on the one diagram, they can be used to compare frequency distributions

Example

The frequency polygons in Fig 25 show a comparison between frequency of passes and frequency of failures in an area examination for pupils of various grading quotients The frequency table is

Grading Quotient	*Mid-interval Value*	*Pass Frequency*	*Fail Frequency*
91– 95	93	40	63
96–100	98	67	24
101–105	103	75	28
106–110	108	50	8
111–115	113	25	2
116–120	118	16	1

Note. The end lines AB, AL, GH, QH are here used only to close the polygons

Exercise

Draw a histogram to illustrate Example B (page 93) and complete the corresponding frequency polygon

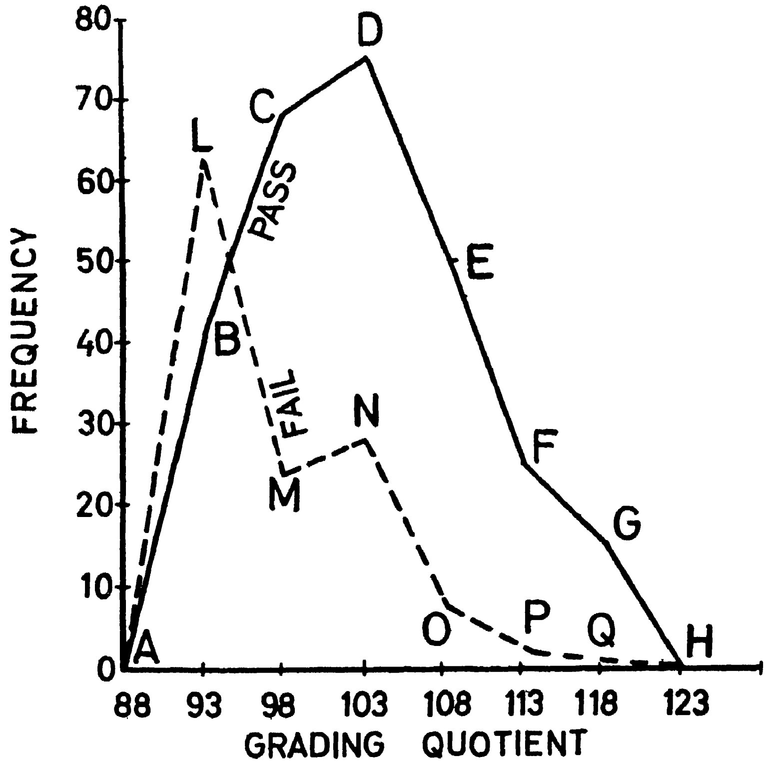


Fig 25

Continuous and Discrete Variation

In Example A the boys' heights could take any values intermediate to those shown. For example, one boy could be $57\frac{7}{8}$ inches, another $57\frac{15}{16}$ inches, another $57\frac{31}{32}$ inches, and so on These heights are said to vary **continuously.** The temperatures in Example B also show **continuous variation.** Where there is a definite break between one value and the next, as in the "number of sums correct" or where share prices on the Stock Exchange rise by a series of jumps, the variation is said to be **discrete.**

Exercises on Chapter 9

1. The following are cricket scores made by members of a team in 60 individual innings

10	1	14	0	11	0	25	27	0	34	17	16
2	12	3	1	33	4	16	18	5	11	12	31
0	1	0	9	13	8	5	6	10	7	5	7
6	3	5	16	8	17	9	0	7	0	3	4
7	21	5	6	2	24	7	5	22	2	6	8

(*a*) Using tally-marks make a grouped frequency distribution with an interval of 5 runs, for example, 0–4, 5–9, etc
(*b*) State whether this distribution shows discrete or continuous variation
(*c*) Construct the histogram and find the modal class

2. The following is the frequency distribution of weights of 100 women

Weight (lb)	*Frequency*	*Weight* (lb)	*Frequency*
80– 89	2	140–149	16
90– 99	3	150–159	10
100–109	6	160–169	5
110–119	10	170–179	4
120–129	19	180–189	2
130–139	23		

Construct the histogram and the frequency polygon

3. The table shows the weekly profit of 120 shops

Weekly Profit (£'s)	0–4	5–9	10–14	15–19	20–24	25–29	30–34	35–39	40–44	45–49
No of Shops	2	5	10	17	28	26	15	7	6	4

(*a*) Construct the histogram and frequency polygon
(*b*) If the sketch (Fig 26) indicates how to obtain the approximate position of the mode from the class intervals with the three greatest frequencies, find its value from your histogram

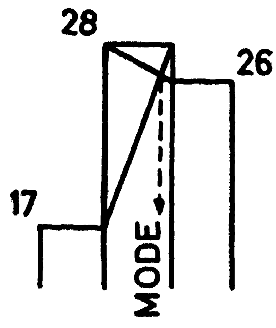


Fig 26

4. The times over 100 yards of 130 sprinters is given in the table

Time (sec)	10 1	10 2	10 3	10 4	10 5	10 6	10 7	10 8	10 9	11	11 1	11 2
No of Men	1	4	7	13	26	34	18	10	8	4	3	2

Draw the histogram and frequency polygon.

5. The table gives the number of deaths occurring over one year in each age group in a village of population 8600.

Age group	0–5	5–15	15–25	25–35	35–45	45–55	55–65	65–75
Deaths	8	3	3	4	6	11	14	15

Draw the histogram and note its U-shape As the class interval 0–5 is only half the other class intervals, to make the area proportional to the frequency the height of this rectangle will have to be doubled

6. *Frequency* To fix a piece-work rate a firm checked the number of articles completed per worker per day The observations made over 12 days were as follows

Worker

A	58	60	61	62	60	59	55	59	63	63	61	61
B	59	63	64	61	62	57	57	65	64	60	58	55
C	61	68	67	68	67	60	62	66	66	67	65	63
D	55	60	62	58	58	56	56	61	63	59	57	57
E	63	64	60	59	56	55	60	65	60	58	57	55
F	67	68	68	67	68	66	66	67	65	67	66	65
G	60	60	61	62	60	60	59	59	60	58	58	57
H	64	65	64	63	62	61	61	62	61	60	59	58

(*a*) Arrange these results in the form of a frequency table with headings (i) *No of Articles*, and (ii) *Frequency*.

(*b*) Draw the histogram and the frequency polygon

(*c*) Which is the modal number of articles completed?

(*d*) If this modal number should give a basic pay rate of £1 5s 0d per day, what should be paid per article?

Chapter 10

STATISTICS (3)

Averages and Dispersion

Groups of values can more easily be understood if they are tabulated in the logical order of a frequency distribution and then converted to graphical form But one single value which summarizes the general size of a group of values, and which the mind grasps most easily, is the **average** of the group We study three kinds of averages or means

(1) The Arithmetic Average or Arithmetic Mean
(2) The Median.
(3) The Mode (see Chapter 9)

THE ARITHMETIC AVERAGE OR ARITHMETIC MEAN

A **The Arithmetic Mean** of a set of quantities is the sum of the quantities divided by their number If the temperatures recorded hourly from 9 a m to 6 p m are 48 6°, 49 9°, 52 4°, 56 8°, 61 3°, 62 2°, 63 4°, 61 8°, 61 4°, 60 2°, the sum is 578 0° for 10 readings and the

$$\text{Arithmetic Mean} = \frac{578\ 0^\circ}{10} = 57\ 8^\circ$$

B THE ARITHMETIC MEAN· CALCULATION FROM AN ASSUMED MEAN

Example

The results of an experiment carried out by eight pupils on the density of mercury in grammes per cubic centimetre gave the following results 13 1, 13 75, 13 9, 13 2, 12 9, 13 7, 13 15, 13 8

We choose some central value as an **assumed mean.** Let the assumed mean be 13 5 The deviations (or differences) from the assumed mean + (greater than) and − (less than) are calculated

Deviations

−	+
0 4	0 25
0 3	0 4
0 6	0 2
0 35	0 3
−1 65	+1 15

The algebraic sum $= -0\ 5$ Average deviation $= \dfrac{-0\ 5}{8} = -0\ 06$ (approx)

$$\text{Mean density} = (13{\cdot}5 - 0\ 06) \text{ grammes per cubic centimetre}$$
$$= 13\ 44 \text{ grammes per cubic centimetre}$$

Exercise

Rewrite the calculation using 13 3 as the assumed mean

C WEIGHTED MEAN

If a shopkeeper sells equal weights of three different brands of tea costing respectively 6s , 6s 9d , and 7s per pound then

$$\text{Mean S P per pound} = \frac{6\text{s} + 6\text{s } 9\text{d.} + 7\text{s.}}{3}$$
$$= \frac{19\text{s. } 9\text{d.}}{3}$$
$$= 6\text{s } 7\text{d}$$

If, however, his sales of the three brands were 5 lb, 12 lb, and 7 lb, respectively, the prices must be weighted according to these figures

$$\text{Mean S.P per pound} = \frac{\text{Total S P.}}{\text{Total No of lb}}$$
$$= \frac{5 \times 6\text{s} + 12 \times 6\text{s. } 9\text{d} + 7 \times 7\text{s.}}{5 + 12 + 7}$$
$$= \frac{160\text{s}}{24}$$
$$= 6\text{s } 8\text{d}$$

The result is called the **weighted mean** of the prices In examples on means worked from frequency tables the frequencies take the place of the weights

Use of symbols

In the examples that follow

X = a single score or value

f = frequency

Xf = product of score and frequency

x = the deviation of a score from an assumed mean

Σ stands for "the sum of," for example Σf = sum of the frequencies

D. THE ARITHMETIC MEAN FROM A FREQUENCY TABLE

Example

The following are the scores over 18 holes of 62 golfers Calculate the mean score to the nearest whole number

First Method *the sum and division method*

Score X	Frequency f	Xf
68	1	68
69	3	207
70	2	140
71	6	426
72	7	504
73	10	730
74	15	1110
75	9	675
76	5	380
77	4	308
	$\Sigma f = 62$	$\Sigma Xf = 4548$

$$\text{Mean} = \frac{\Sigma Xf}{\Sigma f} = \frac{4548}{62} = 73\ 4$$

Mean score $= 73$

Second Method *using an assumed mean of* 74

Score X	Frequency f	Score Deviation from 74 x	xf
68	1	-6	-6
69	3	-5	-15
70	2	-4	-8
71	6	-3	-18
72	7	-2	-14
73	10	-1	-10
74	15	0	0
75	9	$+1$	$+9$
76	5	$+2$	$+10$
77	4	$+3$	$+12$
	$\Sigma f = 62$		-71 $\quad +31$
			$\Sigma xf = -40$

$$\begin{aligned} \text{Mean} &= 74 - \tfrac{40}{62} \\ &= 74 - 0\ 6 \\ &= 73\ 4 \end{aligned}$$

Mean score $= 73$

The second method is preferable because it reduces the amount of multiplication, but care must be taken of the signs.

Exercise

Repeat the calculation made in the second method, using an assumed mean of 71

E. THE MEAN USING MID-INTERVAL VALUES (Xm) AND AN ASSUMED MEAN

To obtain an approximate mean we assume that the average of the values in any interval occurs at the mid-point of that interval The larger the frequency in each interval the more nearly does this approach the truth.

Example

The following are the test scores of 112 children. Calculate the mean score from an assumed mean of 67.

Score X	Mid-interval Score Xm	Frequency f	Deviation from 67 in Class Intervals d	df
90–94	92	3	+5	+15
85–89	87	5	+4	+20
80–84	82	11	+3	+33
75–79	77	17	+2	+34
70–74	72	32	+1	+32
65–69	67	24	0	0
60–64	62	15	−1	−15
55–59	57	4	−2	− 8
50–54	52	1	−3	−3
		$\Sigma f = 112$		+134 −26
				$\Sigma df = +108$

$$\text{Mean} = 67 + \frac{5\Sigma df}{\Sigma f}$$
$$= 67 + \frac{5 \times 108}{112}$$
$$= 67 + 4.8$$
$$= 71.8$$

The Median

A. USE OF THE MEDIAN

When a series of values has been arranged in numerical order, the median value is so placed that there are as many values above it as below it.

Example A An odd number of values
Temperature (°F)

68
66
65
62
57 Median = 57
56
55
53
49

Example B An even number of values
Price per ton (£'s)

35
33
32
29
26 Median = $27\frac{1}{2}$
25
24
22

Rule If there are n values the position of the median is $\dfrac{n+1}{2}$ *

* If the number of values is large the difference between the $\frac{n}{2}$th and the $\left(\frac{n+1}{2}\right)$th value is negligible, and in this case the median is taken as the $\left(\frac{n}{2}\right)$th value.

In Example A $\frac{n+1}{2} = \frac{9+1}{2} = 5$ The fifth value is the median

In Example B $\frac{n+1}{2} = \frac{8+1}{2} = 4\frac{1}{2}$ The value mid-way between the fourth and fifth values is the median

The median, like the arithmetic mean, can be an actual value in the series, as in Example A, or a value which is not a member of the series but falls between two values, as in Example B. The median divides the histogram into two parts of equal area.

B. THE MEDIAN FROM A FREQUENCY TABLE

This is found by using an additional column for **cumulative frequency.** The example shows the number of caps of various sizes worn by 178 boys in a school The total number of caps of sizes $6\frac{3}{8}$ and $6\frac{1}{2}$ is $2 + 14 = 16$, etc

Size of Cap	*Frequency*	*Cumulative Frequency*
$6\frac{3}{8}$	2	2
$6\frac{1}{2}$	14	16
$6\frac{5}{8}$	37	53
$6\frac{3}{4}$	53	106
$6\frac{7}{8}$	41	147
7	26	173
$7\frac{1}{8}$	5	178

The median size corresponds to the value $\frac{179}{2} = 89\frac{1}{2}$ and is therefore $6\frac{3}{4}$

The median can also be found from the cumulative frequency curve (or **ogive**) shown in Fig. 27 The median value is beyond size $6\frac{5}{8}$ and does not exceed size $6\frac{3}{4}$ Therefore, it falls in the size $6\frac{3}{4}$

Advantages and Disadvantages of the Mean, the Median, and the Mode

(a) *The Mean*

Advantages

(i) It is easy to calculate and can be determined exactly
(ii) It makes use of all the values available
(iii) It can be used in arithmetical calculations

Disadvantages

(i) It may be unduly affected by abnormal values unless
(a) The total number of values is very large,
or (b) The abnormal values are excluded before the average is calculated
(ii) It may not be an actual value, for example, the calculated average number of eggs produced per bird per week on a poultry farm might be 4 23

(*b*) *The Median*

Advantages·

(i) It is simple to understand.
(ii) It often represents an actual value in the group
(iii) It is characteristic of the normal group and can be studied For example, in a regiment the soldier of median weight can be examined.
(iv) It is unaffected by extreme values.

Disadvantages

(i) In small groups it may not be characteristic of the group
(ii) It cannot be used for arithmetical calculations.
(iii) It cannot be located accurately in grouped distributions involving class intervals

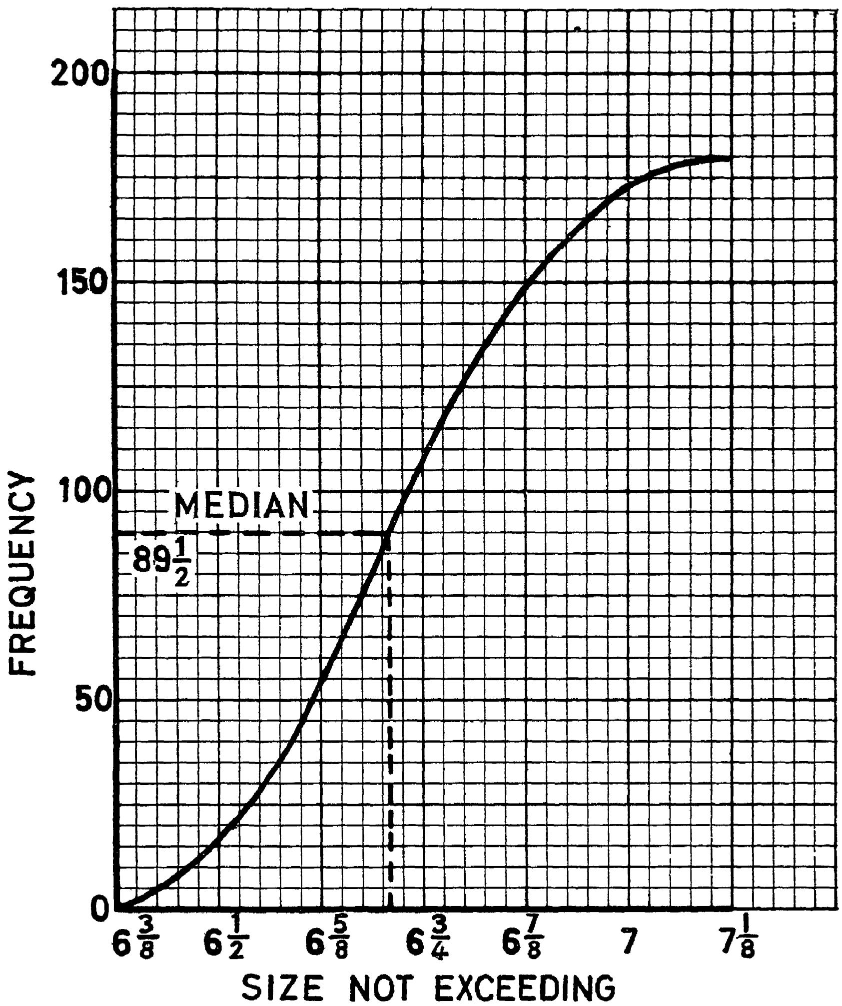


FIG 27

(c) *The Mode*

Advantages

(i) It is simple to understand

(ii) It is unaffected by extreme values

(iii) It is useful for manufacturers of ready-made clothes, etc , to know the modal size of shirt, hat, shoe, and so on

Disadvantages

(i) It is not easy to determine in grouped distributions involving class intervals.

(ii) It cannot be used for arithmetical calculations

Dispersion

While the averages of two or more groups of values are the same, the range or spread of values on either side of the average may differ greatly from group to group Consider the following sets of examination marks

(i)	25	30	35	40	45	50	55	60	65
(ii)	41	42	43	44	45	46	47	48	49
(iii)	18	42	43	44	45	46	47	48	72

The *Mean* of each series = 45.

The *Range* of marks for

(i) is from 25 to 65 = 40 marks,

(ii) is from 41 to 49 = 8 marks,

(iii) is from 18 to 72 = 54 marks

In (i) and (ii) the range shows clearly how the candidates are spread, but in (iii) we see how the range can depend on extreme cases In this case it would be necessary to consider another method of measuring the spread

Consider these three examples

Example (1)

Samples from two manufacturers of wire strands for a suspension cable each have an average diameter of 0 2 inches When the diameters were checked at 15 places the results were·

Sample A	0 21	0 19	0 22	0 22	0 18	0 19	0 20	0 22
Sample B	0 20	0 24	0 25	0 19	0 16	0 24	0 18	0 20

Sample A	0 19	0 20	0 18	0 22	0 18	0 19	0 21
Sample B	0 17	0 19	0 24	0 20	0 16	0 19	0 20

The dispersion of values from the average is from 0 18 to 0 22 inches in Sample A = 0 04 inches, and from 0 16 to 0 25 inches in Sample B = 0 09 inches.

If there are 10,000 strands in the cable the difference between Sample A and Sample B will be considerable, even if the diameter of the finished cable, calculated via areas, is approximately 100 times the diameter of one strand

Example (2)

Mr Jones had two equal holdings of shares in two different companies giving approximately

the same yield The Stock Exchange prices in shillings for these holdings for nine months were

Stock X	112	$113\frac{1}{2}$	$112\frac{1}{2}$	110	109	109	$111\frac{1}{4}$	$111\frac{1}{2}$	$112\frac{1}{2}$
Stock Y	115	118	107	$120\frac{3}{4}$	$121\frac{1}{2}$	104	99	115	101

The average price for each is $111\frac{1}{4}$ but the dispersion (fluctuation) in Stock X is much less than in Stock Y Other factors being equal, if Mr Jones decides to sell one of the holdings he would probably choose Stock Y

Example (3)

The grading quotients (G Q 's) of 35 boys entering a co-educational school range from 123 to 108 and 35 girls entering the same school range from 122 to 106. The averages of these two groups may be almost equal but the headmaster may decide to divide the boys and girls into two mixed classes of 35 to reduce the spread of measured ability rather than into one class of boys and one of girls.

Exercises on Chapter 10

1. Choosing any suitable assumed mean, find the mean of the following marks

75, 48, 66, 59, 54, 68, 73, 84, 65, 48

Check your answer by making a second calculation from another assumed mean.

2. The mean mark of 4 boys in a test was $9\frac{1}{4}$ The mean mark of 9 girls in the same test was 6 Find the mean mark of all 13 pupils

3. For 2 hours of a journey a car averaged 42 m p.h For the next 4 hours its average was 36 m p h. Show that its average speed over the whole journey was 38 m.p h

4. (*a*) Find by inspection the mean of the first nine whole numbers (1, 2 .   9). *From your answer* calculate their sum. Check by addition.

(*b*) Find similarly the mean of the first eight whole numbers.

(*c*) Write down a simple formula for finding the average of the numbers 1 to 3, 1 to 5; and so on. Does this formula apply when there are an even number of terms?

5. In 8 completed innings a cricketer scores:

42, 23, 51, 18, 37, 26, 20, 47.

Working from an assumed average of 30, calculate his batting average

6. The July rainfall in inches for the last 12 years was

1 43, 1 74, 2 31, 0 87, 1 26, 2 05, 1 88, 0 09, 4 68, 1 65, 2·22, 1 3.

(*a*) Using an assumed mean of 1 7 inches, calculate the mean July rainfall for the 12 years

(*b*) Arrange the figures in numerical order and find the median

(*c*) Which of these two measures would you use in predicting the most likely rainfall for July?

7. The internal monthly deliveries of Derv fuel for 1960 in thousands of tons were

Jan	193 7	Apr	202 4	July	215 6	Oct	227 5
Feb	205 6	May	216 4	Aug	213 3	Nov.	230 4
Mar	220 0	June	213 7	Sept	225 3	Dec.	218 2

Calculate, using an arbitrary origin (assumed mean) of 212.0, the average monthly delivery. Check from any other origin.

8. The weights assigned to various subjects in a technical school were

English (E)	2	History (H)	1	Geography (G)	1
Maths (M)	3	Technical Subjects (T)	4	Science (S)	1

Calculate the average mark scored by each of the following students allowing for correct weighting.

	E	H	G	M	T	S
Brown	72	66	79	53	49	54
Smith	52	50	57	84	72	68
Jones	65	62	64	68	60	63

9. The wage rates per hour of 90 male employees in a firm are shown in the table.

Wage per hour (pence)	45	48	51	54	57	60	63
No. of men	5	12	20	24	16	9	4

(*a*) Find the mean wage rate per hour working from 54 pence as origin.
(*b*) Compile a cumulative frequency table and find the median rate.

10. The following table gives the grouped scores of 200 children in an attainment test.

Score	*Frequency*	*Score*	*Frequency*
90–99	2	40–49	43
80–89	7	30–39	24
70–79	24	20–29	6
60–69	39	10–19	3
50–59	52		

Using mid-interval scores of $94\frac{1}{2}$, $84\frac{1}{2}$, etc., calculate the mean of the group from an assumed mean of $54\frac{1}{2}$.

11. The ages of the 125 entrants for a boys' golf championship were

Age	11	12	13	14	15	16	17
No. of boys	1	3	7	12	19	35	48

(*a*) Calculate the mean age from an assumed mean of 16.
(*b*) Find the median age from a cumulative frequency table.
(*c*) Write down the modal age.

12. The following table shows the wage distribution in a certain factory.

Weekly Wage (£'s)	*No. of Employees*
4.5– 5.5	12
5.5– 6.5	19
6.5– 7.5	31
7.5– 8.5	54
8.5– 9.5	108
9.5–10.5	154
10.5–11.5	96
11.5–12.5	75
12.5–13.5	10
13.5–14.5	4

Find

(*a*) The mean weekly wage
(*b*) The median wage group
(*c*) The modal wage group

13. The histogram (Fig. 28) shows the distribution of a shop's sales of men's shirts of various sizes

Find

(*a*) The total number of shirts sold
(*b*) The mean size of the shirts sold
(*c*) The median size
(*d*) The modal size

14. The following table gives the spread of candidates in an examination

Mark	*Frequency*	*Mark*	*Frequency*
95–99	5	65–69	17
90–94	3	60–64	24
85–89	5	55–59	19
80–84	11	50–54	8
75–79	14	45–49	5
70–74	15	40–44	4

(*a*) Using deviations from mid-interval values, calculate the mean mark
(*b*) State in which class interval the median mark lies

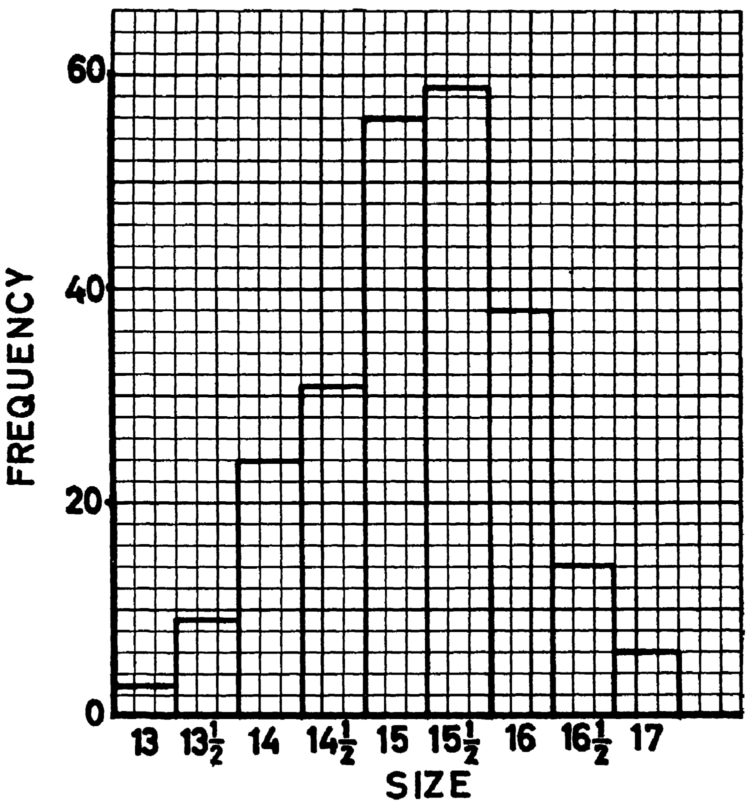


Fig 28

ANSWERS

(Where answers are obvious, these have been omitted)

CHAPTER 1

Exercises A (p 6)

1. E (6, 0), F (4, 5), G (0, 3), H (−2, 8). K (−7, 4), L (−9, 0), M (−3, −2), N (−5, −8), P (0, −7), R (3, −5), S (7, −3) (*a*) (i) first, (ii) third (*b*) fourth (*c*) second (*d*) 0 (*e*) 0

2. KL = 4 13 in, MN = 3 61 in, PQ = 5 in, RS = 6 42 in, TV = 2 22 in

3. Middle points are $(2, -\frac{1}{2})$, $(0, -1\frac{1}{2})$, $(-2, 1\frac{1}{2})$, $(\frac{1}{2}, \frac{3}{4})$.

4. (*e*) $(2, -\frac{1}{2})$

7. AB = BC = 3 6 in

8. Distance from origin = 5 units A circle

10. Perimeter = 12 3 in

Exercises D (p 11)

1. 12, 7, 2, 27, −8, −13, −18

2. 5, 3, $1\frac{2}{3}$, 1, $\frac{1}{3}$, −1, −3

3. 13, 12, 11, 10, 9, 6, −3

4. $y = -12, -7, -2, 3, 8, 13$

5. $y = -4\frac{1}{2}, -1\frac{1}{2}, 0, 1\frac{1}{2}, 3, 4\frac{1}{2}, 7\frac{1}{2}$.

6. --8, −4, 2, 10, 20

7. −6, −12, −10, −6, 0, 18, 44.

8. −77, −17, −5, 7, 67

9. $\frac{1}{4}$, $\frac{1}{4}$, $2\frac{1}{4}$, $6\frac{1}{4}$

EXERCISES ON CHAPTER 2 (p 18)

3. $x = 3, y = 1$.

4. 35 square units

5. Values of $4x - 5$ increase by 8 Graph is straight line going upward from left to right

6. Values of $17 - 2x$ decrease by 6 Graph is a straight line going downward from left to right

7. (*a*) −9, −1, 1, −8 (*b*) −2, 1, 3, $-1\frac{1}{2}$, $3\frac{1}{2}$, $3\frac{3}{4}$

8. $18\frac{3}{4}$ square units

EXERCISES ON CHAPTER 3 (p 26)

1. (i) $x = 3, y = 5$ (ii) $x = 2\frac{1}{2}, y = 5\frac{1}{2}$. (iii) $x = 7\frac{1}{2}, y = 5$ (iv) $x = 1, y = -1$ (v) $x = -3, y = -4$ (vi) $x = 4\frac{1}{3}, y = -7\frac{1}{3}$

2. $x = 3, y = -1$

3. $x = 2, y = 5$, $x = 4, y = 1$, $x = -4, y = 2$ Area of $\triangle$ = 7 5 square units (approx).

4. $a = 5\ 2$, $b = 0\ 3$. (i) 5 65. (ii) 5 88 (iii) 7 45 (iv) 5 2 (v) 0 4 (vi) 1 1. (vii) 5 4. (viii) 7 3

5. $k =$ 0 3, $a =$ 101. (i) 107 (ii) 110 (iii) 113 3 (iv) 16 (v) 28 3. (vi) 42 5.
6. $r =$ 0 52, $s =$ 21 55 (i) 21 55, unstretched length of the spring. (ii) 22 7. (iii) 23·83. (iv) 24 06.
7. $k =$ 3 25, $l =$ 55 (i) 172 (ii) 380 (iii) 75
8. $f =$ 598, $g = -$7 9 (i) 590, 527, 400, 211
9. $k = 3700$

Exercises on Chapter 4 (p 34)

1. (i) $x = 2, 1, 5$
(ii) $x = 4, -18, -2$
(iii) $x = 2\frac{1}{2}, -13\frac{1}{4}, -7$
(iv) $x = -\frac{1}{2}, -4\frac{1}{2}, -4$
(v) $x = -3, -6, 3$
(vi) $x = 1, 3, 4$
(vii) $x = \frac{1}{2}, 4\frac{3}{4}, 5$
(viii) $x =$ 1 7, 2 91, 5 8
(ix) $x = \frac{2}{3}, \frac{1}{18}, \frac{1}{2}$
(x) $x = -\frac{3}{8}, -3\frac{9}{64}, -3$
5. (i) $x = 0, 0, 0$ (ii) $x = -$1 5, 0, 2 25 (iii) $x =$ 2 5, 0, 6 25
6. (i) $x = -1, 0, 1$ (ii) $x = -1$, 6 5, 7 5 (iii) $x = -1$, $-$2 25, $-$1 25
7. $1\frac{1}{4}, -1, -\frac{1}{5}, \frac{7}{8}, -\frac{3}{4}, \frac{1}{12}, 1\frac{1}{2}, -2$

Exercises on Chapter 5 (p. 47)

1. (*a*) to (*f*) (i) Turning values $-3, -3\frac{1}{4}, 9\frac{1}{4}, 6, -10\frac{1}{8}, 20\frac{1}{2}$
(iv) (*a*) Increasing ranges -1 to 1, $2\frac{1}{2}$ to 6, -3 to $1\frac{1}{2}$, -5 to -1, $1\frac{3}{4}$ to 5, -1 to $2\frac{1}{2}$.
(*b*) Decreasing ranges -5 to -1, -1 to $2\frac{1}{2}$, $1\frac{1}{2}$ to 6, -1 to 3, -1 to $1\frac{3}{4}$, $2\frac{1}{2}$ to 5
(v) (*a*) $-$3 73 to 0 27, 0 7 to 4 3, $-$0 5 to 4
(*b*) $-$1 54 to 4 54, $-$3 45 to 1 45, $-$0 13 to 5 13
2. (*a*) $-3\frac{1}{4}$ (*b*) $-$3 57 to -3 and 0 to 0 57
3. (*a*) $7\frac{1}{8}$. (*b*) $-$1 64 to 2 14

Chapter 6

Example (1) (p. 48)

1. (*c*) 3, 0 **2.** (*b*) 4, -1 **3.** (*b*) 2, 1 **4.** 4 5, $-$1 5 **5.** 0 5, 2 5. **6.** (*a*) $1\frac{1}{2}$, (*b*) the minimum value of $x^2 - 3x$ is $-2\frac{1}{4}$

Example (2) (p 50)

1. (*b*) $-\frac{1}{2}, 3$ **2.** (*b*) $-1, 3\frac{1}{2}$ **3.** (*a*) $-1\frac{1}{2}, 4$. **4.** $0, 2\frac{1}{2}$. **5.** (*a*) $-6\frac{1}{8}$ at $x = 1\frac{1}{4}$, (*b*) the lowest point

Example (3) (p 52)

1. $-$2 56, 1 56 **2.** $-3, 2$ **3.** $-2, 1$ **4.** (*a*) two, two, one **5.** $-4\frac{1}{4}$ **6.** (*a*) (*b*) (*c*) $-4\frac{1}{4}$

Example (4) (p 54)

2. $-$0 45, 4 45 **3.** $-$0 45, 4 45 **4.** $-1, 5$. **5.** 1, 3 **6.** 2 **7.** no **8.** (*a*) (*b*) (*c*) 6

Example (5) (p 56)

1. $-$1 21, 0 21 **2.** $-$3 5, 2 5 **3.** $-$2 23, 1 23 **4.** $-1, 0$ **5.** $-$0 5. **6.** $-3, 2$ **7.** $-2, 1$
8. $-$1 82, 0 82 **9.** $-$1 5, 0 5 **10.** $-$3 2, 2 2 **11.** $-$2 5, 1 5. **12.** $y = -x + 1$
13. $y = -2x - 2$

Example (6) (p 58)

1. (*a*) −1, 2 5 (*b*) −1 61, 3 11 (*c*) −0 61, 2 11 (*d*) 0 75 (*e*) −2, 3 5 (*f*) −0 93, 2·43 (*g*) −0 28, 1 78, (*h*) −1 56, 3 06, (*i*) 0, 1 5, (*j*) −1 14, 2 64

2. (*a*) $k < 6\frac{1}{8}$ (*b*) $k = 6\frac{1}{8}$ (*c*) $k > 6\frac{1}{8}$

3. (*a*) 3 (*b*) $x = -0\ 5$

4. (*a*) −1 28, 0 78 (*b*) −0 58, 2 58 (*c*) −0 89, 3 39

Example (7) (p. 60)

1. (*a*) −1 12, 0 72 (*b*) −2, 1 6 (*c*) −0 73 or 0 33 (*d*) −1 76 or 1 36 (*e*) −1, 0 6 (*f*) −0 2 (twice) (*g*) −1 22, 0 82 (*h*) −1 94, 1 54

2. (*a*) $t > -4\ 2$ (*b*) $t = -4\ 2$ (*c*) $t < -4\ 2$

3. (*a*) −1 63, 1 23 (*b*) −0 94, 0 54 (*c*) −1 27, 0 87

4. $t = 3\ 6$, $x = -1\ 45$

6. (*a*) −1 11, 0 91 (*b*) −1 47, 0 27. (*c*) −0 36, 0 56

Exercises on Chapter 6 (p 66)

1. (i) 0, 2. (ii) −4, 6 (iii) −3, 5 (iv) −1 24, 3 24 (v) 1 (twice) (vi) 0 29, 1 71.

2. (i) 3, 4 (ii) 1, 6 (iii) 0, 7. (iv) 1 21, 5 79 (v) 2, 5 (vi) $1\frac{1}{2}$ (twice)

3. (i) −4, 2 5. (ii) −3 5, 2 (iii) −2 89, 1 39 (iv) −2 5, 1 (v) −0 75 (twice)

4. (i) −3, 1 (ii) no solution (iii) −1 58, −0 42 (iv) −1 (twice). (v) 0, −2. (vi) −3, 1 (vii) 0, −2

5. (*a*) (i) −0 75, 2 (ii) −2, 3 25 (iii) −1 59, 2 84 (iv) −0 5, 1 75 (v) 0 25, 1 (vi) no solution. (*b*) $4x^2 - 5x = -1\frac{9}{16}$ (*c*) $x = 0\ 625$ (twice)

6. (*a*) (i) −3, 3 2 (ii) −1 9, 2 1. (iii) no solution (iv) no solution (v) 0 1 (twice) (vi) −1 32, 1 52

7. (i) −1, 2. (ii) 1, 3 (iii) −2, 1 5

8 (*a*) (i) −1, 3 5 (ii) −2, 2 5 (iii) −3 59, 2 09 (iv) −1 45, 3 45

9. Solutions not necessary

10. (i) 1 5, 4. (ii) 1 04, 5 76 (iii) 4 9

Chapter 7

Example (1) (p 68)

A **4.** −17 6, −9 3, −4 1, −0 3, 0 7, 2 7, 6 9, 15 6.

5. −2 76, −2 29, −1 52, −1 26, 1 14, 2, 2 12, 2 67, 2 8, 2 92

Example (2) (p 70)

6. −1 84, 0 91 (twice)

7. (i) −1 75, 0 43, 1 3 (ii) −1 5, −0 15, 1 65 (iii) −1 3, −0 41, 1 75 (iv) 2 58. (v) −1 92 (vi) 0, ±1 58

Example (3) (p 72)

2. (i) 13 1, 1 5. (ii) −1 1, −1 5

3. (i) −2, −1, 3 (ii) −2 4, −0 45, 2 8 (iii) −1 95, −1 16, 3 03 (iv) 3 15 (v) −3 4 (vi) −2 9, 0 67, 2 24.

4. (*a*) Between −1 1 and 13 1 (*b*) −1 1, 13 1 (*c*) $k < -1\ 1$ or $> 13\ 1$

Example (4) (p. 74)

2. (i) −2 to 0, and $x > 1$ (ii) 0 to 1 (iii) −1 22 to 0 55

3. Max. = 2 1, $x = -1\ 22$ Min = −0 65, $x = 0\ 55$

4. (i) -2 16 (ii) 1 55. (iii) -1 91, -0 23, 1 14. (iv) -1 73, -0 56, 1 29.
5. (*c*) 3, -12

Example (5) (p. 76)

2. -2 73, 0 73, 2
3. (*a*) $x^3 - 6x \pm 5\ 5 = 0$ (*b*) ± 1 4 (twice), ± 2 83
4. (i) One (ii) One. (iii) Three
5. (i) -1 5, -0 4, 1 87 (ii) -1 6, 0 7, 1 (iii) 2 75. (iv) 0 33 (v) 0 45. (vi) -1 12.
6. One

Example (6) (p 80)

7. (i) -6, 5 (ii) -3 3, 6 (iii) 1, 7 5

Exercises on Chapter 7 (p 80)

1. -1, -0 6, 1 62.
2. (i) -1 93, -0 53, 2 45 (ii) -1 73, -0 8, 2 52 (iii) -2 13, -0 21, 2 32 (iv) -2 42, 0 42, 2. (v) 2 72 (vi) -2 34, 0 21, 2 12.
3. (i) 2 86 (ii) -0 3
4. (*a*) (i) $-2\frac{1}{2}$ to -1, 1 to $2\frac{1}{2}$ (ii) $-2\frac{1}{2}$ to -2, 0 55 to 1 37 (*b*) 7, $x = -1$; -1, $x = 1$ (*c*) -1 95, 0 55, 1 37.
5. (*a*) -3 84, -0 07 (*b*) 21 4, -23 4 (*c*) -1 (*d*) (i) -3 84, -0 07, 3 88. (ii) -3 78, -0 2, 3 97 (iii) -3 94, 0 13, 3 8
6. (i) -3, 0 39, 2 62 (ii) -2 69, -0 25, 2 95 (iii) -3 06, 0 53, 2 54 (iv) $-2{\cdot}84$, 0 45, 2 4
7. (*a*) (i) -1, 0 67 (ii) -1 57, -0 24, 1 31 (iii) 0, 4 6 (*b*) -1 5, 1 62 (*c*) 1 82.
8. Square of side $1\frac{1}{3}$ in
10. (*a*) $x = -1, y = 0$ (*b*) $x = 1, y = 0$.
11. $x = 2$ or $x = 4$.
12. (*b*) Strips 9 in by 4 in

Chapter 8

Example (2) (p 88)

4. $14\frac{3}{4}\%$, $18\frac{3}{4}\%$

Example (3) (p 88)

1. 36 25 **2.** £144 **3.** £8 £20 **6.** 144°, 120°
7. £24 per annum

Exercises on Chapter 8 (p 90)

5. (*a*) (i) 27° (ii) 65° (*b*) (i) $2\frac{1}{2}\%$ (ii) 5% (iii) $11\frac{7}{18}\%$

Exercises on Chapter 9 (p. 96)

1. (*a*) Frequencies 18, 20, 8, 6, 3, 2, 3 (*c*) Modal class 5–9.
3. (*b*) £23 15s
6. (*a*) 5, 3, 6, 8, 8, 14, 10, 7, 7, 5, 6, 5, 7, 5 (*c*) 60. (*d*) Fivepence

Exercises on Chapter 10 (p 105)

1. 64 **2.** 7 **4.** (*c*) $\dfrac{n+1}{2}$ **5.** $30 + \frac{24}{8} = 33$ **6.** (*a*) 1 79 in (*b*) 1 695 in.

7. $212 + \dfrac{38\ 1}{12} = 215\ 2$.

8. 58 2, 68 3, 63 6.
9. (*a*) 53 57 pence (*b*) 54 pence
10. 54 25
11. (*a*) 15 years 9 months (15 74) (*b*) 16 years (*c*) 17 years
12. (*a*) £9 14s 7d (*b*) (*c*) £9 5s –£10 5s.
13. (*a*) 240 (*b*) 15 173 (*c*) 15 (*d*) $15\frac{1}{2}$
14. (*a*) 66 9 (*b*) 65–69

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